The Halting Problem is Unsolvable

**Theorem 1**
The halting problem is unsolvable, i.e. the language \textbf{Halting} is not recursive.

Proof: Proof by contradiction: Assume that the language \textbf{Halting} \((H)\) is recursive and is decided by \(\text{TM } T_H \text{ s.t. } L(T_H) = H\).

We need a function \(f\) s.t. \(x \in SA\) if and only if \(f(x) \in H\). If one can decide \(H\) then one can decide \(SA\) (build \(T_{SA}\) in terms of \(T_H\)) \(\Rightarrow\) contradiction.

Let \(T_{SA} = T_1T_H\), where \(T_1\) is a TM with input alphabet \(\Sigma = \{x, y\}\) that transforms a tape of the form \(\Delta w\) to the tape \(\Delta we(w)\). The construction of \(T_1\) is relatively straightforward and not reproduced here.

If \(w \in SA\), then \(w = e(T)\) for some TM \(T\), where \(T\) accepts \(w\). In this case \(T_{SA}\) halts on input \(w\) with output 1, since the string \(we(w) = e(T)e(w)\) is in \textbf{Halting} and \(T_H\) decides \textbf{Halting}.

If \(T_{SA}\) halts with output 1 on input string \(w\), then \(we(w) \in \textbf{Halting}\). This means that for some TM \(T\) and some \(w, T\) accepts \(w\). Since \(w = e(T), w \in SA\).

If \textbf{Halting} were recursive then we can construct the TM \(T_{SA}\) which decides \(SA\). Since we know that \(SA\) is not recursive, we conclude that neither is \textbf{Halting}. \(\diamond\)