The Existence of Languages that are not R.E.

Theorem 1
Let $\Sigma = \{x, y\}$, there are languages over Σ that are NOT recursively enumerable.
Proof: We begin by showing that the set $\Re E$ of recursively enumerable languages over Σ is countable.

Every recursively enumerable language $L \subseteq \Sigma^*$ is accepted by a TM T_L. Let Θ be the set of all such TM's T_L. The encoding that we developed in discussing a Universal Turing machine e is a one-to-one function from Θ to Σ^*.

The observation that the encoding function e from Θ to Σ^* is based on the restriction that all of the states in all of the TM's in Θ are drawn from the same infinite set of states $\aleph$, and that all of the tape alphabets in all of the TM's in Θ are drawn from the same infinite set of tape symbols $\Im$. (For $T_L = (Q, \Sigma, \Gamma, q_0, \delta)$, $Q \subseteq \aleph$, $\Gamma \subseteq \Im$.)

This one-to-one relationship gives us:

$$T : \Re E \rightarrow \Theta$$

$$e : \Theta \rightarrow \Sigma^*$$

and therefore $e \circ T : \Re E \rightarrow \Sigma^*$.

We know that the function e is one-to-one. The function T is also one-to-one, since each TM accepts only one language. This implies that $e \circ T$ is also a one-to-one function. This gives us a bijection from the set $\Re E$ to a subset of Σ^*.

We know that Σ^* is a countable set, since every subset of a countable set is a countable set. Therefore the set $\Re E$ of recursively enumerable languages over Σ is a countable set.

The theorem is proved by observing that there are uncountably infinite languages over the same set Σ. $\diamond$

This Theorem tells us that there are not enough Turing machines T (each of which accepts $L \subseteq \Sigma^*$; $L \in 2^{\Sigma^*}$) to accept all of the languages over Σ (2^{Σ^*}); there are not enough strings around to specify all of the required Turing machines.

In fact there are uncountably many languages that are not in $\Re E$.