The Power Set of a Countably Infinite Set is Uncountable

Theorem 1
If S is a countably infinite set, 2^S (the power set) is uncountably infinite.
Proof: We show 2^S is uncountably infinite by showing that $2^{\mathbb{N}}$ is uncountably infinite. (Given the natural bijection that exists between $2^{\mathbb{N}}$ and 2^S –because of the bijection that exists from $\mathbb{N}$ to S– it is sufficient to show that $2^{\mathbb{N}}$ is uncountably infinite.)

Assume that the set $2^{\mathbb{N}}$ is countably infinite.

The subsets of $\mathbb{N}$ can be listed $A_0, A_1, A_2, \ldots$ so that every subset is A_i for some i.

We define another set $A = \{i | i \geq 0$ and $i \notin A_i\}$ which contains those integers i which are not members of their namesake set A_i.

But A is a subset of $\mathbb{N}$, and so $A = A_j$ for some j. But this means

1. If $j \in A$, then $j \notin A$.
2. If $j \notin A$, then $j \in A$.

We have a contradiction, since j must either be in the set A or not in the set. Therefore $2^{\mathbb{N}}$ is not countably infinite. $\diamond$

The Theorem that the power set of a countably infinite set is an uncountable set indicates that the set of all languages over any alphabet Σ, $|\Sigma| \neq 0$ is uncountable. (2^{Σ^*} is an uncountable set.)