On the Probability of Sequences in the Genoise Lottery

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E338
Mémoire de l'académie des sciences de Berlin [21] (1765), 1767, p. 191–230.

Since the establishment of one such lottery in this city, all the arrangements are so generally known that it would be altogether superfluous to give a description of them. Likewise the majority of the questions that one is able to formulate on the probability of the events which take place in this lottery are no longer unknown, seeing that their solution is deduced easily from the principles established in the science of probabilities. But, when one requires the probability of the *sequences* which can be found among the five numbers which one draws each time, the question is so difficult that one encounters the greatest obstacles to arrive at the solution.

Now, one has a *sequence*, when two or more of the five numbers which one draws each time follow themselves immediately according to the natural order of numbers; whence one understands that it must imply a sequence of two or three or four or all five numbers. Thus, when there are among the five drawn numbers, for example, these two: 7 and 8, this is a *sequence of two*; if there are these three numbers 25, 26, 27, these will be a *sequence of three*; and likewise several. One could think that, since there are only 90 numbers in this lottery, it would be convenient to regard these two: 90 and 1 as a sequence of two; but it is more natural to exclude them and to keep solely to the natural order of the numbers.

Now, it is good to render this question more general, and therefore, I will suppose that instead of 90 tickets there are in all n tickets marked with the numbers

$$1, 2, 3, 4, \ldots, n,$$

and that one draws at random some number which is m. This put, one requires what is the probability that there is found among these m drawn numbers either a sequence of two, or one of three, or one of four etc., or at the same time two sequences of two, or one of two and one of three etc., or finally, that there are no sequences found at all. There are therefore several questions that each case furnishes, of which the number will be as much greater than the number m of drawn tickets will be great.

But, in order to arrive to the solution of all these questions, it is absolutely necessary to begin with the case $m = 2$, where one draws only two from n tickets; from there I

*Translated by Richard J. Pulskamp, Department of Mathematics & Computer Science, Xavier University, Cincinnati, OH. November 15, 2009 will pass to that where one draws 3 of them, so that $m = 3$; next to that where $m = 4$ and $m = 5$ and $m = 6$ etc., as far as the difficulties of calculation permit me to push these researches.

PROBLEM 1

1. *The number of tickets marked with the numbers* $1, 2, 3, 4$ *etc. being* $= n$*, when one draws two tickets, what is the probability that there will be a sequence or not?*

SOLUTION

One knows that the number of all possible cases which are able to take place in the two drawn numbers is

$$= \frac{n(n-1)}{1 \cdot 2}$$

where one does not regard the order of these two numbers, such that, for example, the drawn numbers 7 and 10 form the same case as if they had been drawn 10 and 7. In this number of cases, $\frac{n(n-1)}{1\cdot 2}$, they contain those where there is a sequence as well as those where there is not. Now, it is easy to make the denumeration of all cases which contain a sequence, which are: 1,2; 2,3; 3,4; 4,5; etc. up to the last $n-1, n$; of which the number is evidently $= n - 1$. But the probability of any event is expressed by a fraction of which the numerator is the number of cases where this event occurs, and the denominator is the number of all possible cases; whence one gets the probability that the two drawn numbers contain a sequence

$$= \frac{2(n-1)}{n(n-1)} = \frac{2}{n}.$$

Therefore, that it is not in sequence, the probability will be

$$= 1 - \frac{2}{n} = \frac{n-2}{n}.$$

COROLLARY 1

2. Therefore, the number of numbers which follow in their natural order being $= n$, if one draws two of them, so that the number of all the possible cases $= \frac{n(n-1)}{1\cdot 2}$, the number of cases which contain a sequence is

$$= n - 1$$

and the number of cases which do not

$$= \frac{(n-1)(n-2)}{1 \cdot 2}$$

COROLLARY 2

3. And therefore, the probability that the two drawn numbers contain a sequence is

$$= \frac{2(n-1)}{n(n-1)} = \frac{2}{n},$$

and the probability that the two drawn numbers do not give a sequence is

$$= \frac{(n-1)(n-2)}{n(n-1)} = \frac{n-2}{n}.$$

COROLLARY 3

4. Therefore, if the number of tickets, n, being 90 and if one would extract only two, the probability of a sequence would be $\frac{1}{45}$, and that there would not be a sequence $= \frac{44}{45}$. Or indeed, one could wager 1 against 44 that there will be no sequence.

REMARK.

5. It is evident that the number of cases which give a sequence, being added to the number of cases which do not give them, ought to produce the number of all possible cases, which is $= \frac{n(n-1)}{1 \cdot 2}$; and thence I have concluded that, since the number of cases of a sequence would be

$$= n - 1,$$

the number of cases to the contrary ought to be

$$= \frac{(n-1)(n-2)}{1 \cdot 2}$$

But one is able to find also the same number by actual denumeration. Let one suppose one of the drawn numbers is a; and since the other is known to be neither $a-1$ nor $a+1$, it must be one of the others, of which the number is $n-3$, so that each number gives $n-3$ cases, whence the number of all cases would be $= n(n-3)$; but it is necessary to consider that, if one takes for a either the first 1, or the last, n, the number of cases becomes greater by a unit, since in the first case the number $a-1$ and in the other $a+1$ are not excluded. Consequently, the found number $n(n-3)$ must be increased by two, whence it becomes

$$= nn - 3n + 2 = (n-1)(n-2).$$

But here each case is counted twice, since, putting the two drawn numbers a and b, this same case is reported as much for the number a as for the number b; whence I conclude that the number of cases free of sequences is only the half of $(n-1)(n-2)$ and hence

$$= \frac{(n-1)(n-2)}{1 \cdot 2}$$

I have added expressly this operation, in order to better make known the precautions that it is necessary to take in the following.

PROBLEM 2

6. *The number of tickets marked with the natural numbers 1, 2, 3, 4, etc. being some number*$= n$*, if one draws three at random, to find all the probabilities with regard to the sequences.*

SOLUTION

Here there are three cases to develop in regard to the sequences, that I will represent in the following manner:

I. $a,\ a+1,\ a+2$

this which is a sequence of three.

II. $a,\ a+1,\ b,$

this which is a sequence of two, the third number b, being neither $a+2$ nor $a-1$.

III. $a,\ b,\ c,$

where the numbers a, b, c do not comprise a sequence.

These three cases together must produce all the possible cases of which the number is

$$= \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3}$$

Therefore we make the denumeration of all the cases in each of these three species.

For the first, $a,\ a+1,\ a+2$, the denumeration is very easy, since all these cases are

$$(1,2,3),\ (2,3,4),\ (3,4,5) \text{ etc.,}$$

up to the last

$$(n-2, n-1, n),$$

of which the number is $= n-2$; and thus, the probability that a sequence of three take place

$$= \frac{2 \cdot 3}{n(n-1)}.$$

For the second species, $a,\ a+1,\ b$, we have only to consider all the sequences of two, which are in number $n-1$, and to remark that each receives still one of the other numbers with the exception of the four $a-1,\ a,\ a+1,\ a+2$, such that the number of values of b would be $n-4$. But it is necessary to consider that for the first sequence, 1,2, and the last, $n-1, n$, the number of values of b is $n-3$; and thus, the number of all the cases is

$$(n-1)(n-4)+2 = nn-5n+6 = (n-2)(n-3);$$

which number is already correct, since anyone of these cases cannot occur twice. Therefore, the probability that this species occurs is

$$= \frac{2 \cdot 3\,(n-3)}{n(n-1)}.$$

For the third species, a, b, c, taking the number a at will, the two others, b and c, must be taken from the interrupted sequence of numbers

$$1,\ 2,\ 3,\ \ldots\ a-2,\ |\, a+2,\ a+3,\ a+4, \ldots n,$$

where the number of terms of the first part is $= a-2$ and the other $= n-a-1$; but such that b and c do not make a sequence. We suppose that both are taken from the first part, of which the number of terms is $= a-2$; and since the sequence of numbers 1, 2, 3, 4, $\ldots n$ furnish $\frac{(n-1)(n-2)}{1\cdot 2}$ combinations of two without a sequence, the number of these cases is

$$= \frac{(a-3)(a-4)}{1\cdot 2}.$$

In the same way, if both are taken from the other part $a+2,\ a+3,\ \ldots n$, of which the number of terms is $= n-a-1$, the number of cases is

$$= \frac{(n-a-2)(n-a-3)}{1\cdot 2}.$$

Now, if one takes one from the first and the other from the second part, each combination is free of a sequence, and thus the number of cases will be $= (a-2)(n-a-1)$; whence the number of all the cases for each number a will be

$$\frac{(a-3)(a-4)+(n-a-2)(n-a-3)+2(a-2)(n-a-1)}{1\cdot 2},$$

which reduces to

$$\frac{nn-9n+22}{2}.$$

But this denumeration does not take place when the number a is either 1 or 2 or n or $n-1$, which is necessary to consider separately. Therefore taking place for $n-4$ values of a, the number of cases will be

$$= \frac{(n-4)(nn-9n+22)}{2}.$$

Now, the two cases $a=1$ and $a=n$ each give as many cases

$$\frac{(n-3)(n-4)}{2},$$

and the two cases $a=2$ and $a=n-1$ each give

$$\frac{(n-4)(n-5)}{2};$$

therefore, the number of cases which correspond to these four values altogether will be

$$\frac{2(n-3)(n-4)}{2}+\frac{2(n-4)(n-5)}{2}=\frac{2(n-4)(2n-8)}{2}=\frac{(n-4)(4n-16)}{2},$$

which, being added to the preceding number, produces

$$\frac{(n-4)(nn-5n+6)}{2}=\frac{(n-2)(n-3)(n-4)}{1\cdot 2}.$$

Finally, it is necessary to observe that each triple of numbers a, b, c is counted here three times, since each is able to be drawn instead of a, and thus the correct number of all the cases of this third species reduces to

$$\frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3}.$$

Whence the probability that among the three drawn numbers there is no sequence, will be

$$= \frac{(n-2)(n-3)(n-4)}{n(n-1)(n-2)} = \frac{(n-3)(n-4)}{n(n-1)}.$$

COROLLARY 1

7. Therefore having three species to consider, when one draws three from n tickets, which are

$$\text{I. } a, a+1, a+2, \quad \text{II. } a, a+1, b \quad \text{and} \quad \text{III. } a, b, c,$$

the number of cases for each of these species is:

for the first, $a,\ a+1,\ a+2,$

$$n-2,$$

for the second, $a,\ a+1,\ b,$

$$\frac{2(n-2)(n-3)}{1 \cdot 2}$$

for the third, $a,\ b,\ c,$

$$\frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3}.$$

COROLLARY 2

8. Therefore, in order that there is found in the three drawn numbers a sequence of three, $a,\ a+1,\ a+2$, the probability is

$$= \frac{2 \cdot 3}{n(n-1)}.$$

In order that there is found only a sequence of two, the probability is

$$= 2 \cdot \frac{3(n-3)}{n(n-1)},$$

and in order that there is no sequence, the probability is

$$= \frac{(n-3)(n-4)}{n(n-1)}.$$

COROLLARY 3

9. If one requires the cases where there is found at least one sequence of two among the three drawn numbers, the number of favorable cases is

$$= n - 2 + (n-2)(n-3) = (n-2)^2$$

and therefore the probability

$$= \frac{2 \cdot 3(n-2)}{n(n-1)}.$$

REMARK

10. It is evident here that the numbers of cases which occur to each of our three species, being added together, produce the number of all the possible cases, which is

$$= \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3};$$

all as the nature of the question demands, for it since it is in effect

$$n - 2 + \frac{2(n-2)(n-3)}{1 \cdot 2} + \frac{(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3} = \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3};$$

and in the same manner, the sum of the probabilities which correspond to these three species must equal unity, which is the character of a complete certitude.

For this reason, I would have been able to truly dispense with the embarrassing reasoning by which I made the denumeration of the cases of the third species. But I have added it expressly in order to better make visible the correctness, seeing that it bears overtly the impression of the truth, so that it would not seem suspect, when I am obliged to resort to it in the following. Certainly, since I have arrived finally to an expression so simple, one would scarcely doubt that there would be also another route sufficiently simple which leads to the same conclusion, that which merits principally the attention of those who apply themselves to this type of researches.

PROBLEM 3

11. *The number of tickets marked with the natural numbers 1, 2, 3, 4 etc. being* $= n$*, if one draws 4 at random, to find all the probabilities that there are able to occur in regard to the sequences.*

SOLUTION

Among the four drawn numbers, it is necessary to distinguish 5 different species with respect to the sequences, of which the nature ought to be represented in the following manner:

I. $a,\ a+1,\ a+2,\ a+3$; II. $a,\ a+1,\ a+2,\ b$; III. $a,\ a+1,\ b,\ b+1$;
IV. $a,\ a+1,\ b, c$; V. a, b, c, d;

such that the first contains a sequence of 4, the second one of 3, III. two sequences of two, IV. a single sequence of 2 and V. contains no sequence. Therefore it concerns making the denumeration of the cases for each of these species, of which the sum must be equal to the number of all the possible cases, which is

$$= \frac{n(n-1)(n-2)(n-3)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

I. The number of cases where the first species takes place is

$$= n - 3,$$

since these cases are

$$(1, 2, 3, 4),\ (2, 3, 4, 5), \ldots (n-3, n-2, n-1, n),$$

and thus the probability this species occurs will be

$$= \frac{2 \cdot 3 \cdot 4}{n(n-1)(n-2)}.$$

II. For the species $a,\ a+1,\ a+2,\ b$, the number of all the possible sequences of three being $= n-2$ [§ 7], the number b must be taken either from this progression

$$1,\ 2,\ 3,\ \ldots a-2,$$

or from this

$$a+4,\ a+5,\ \ldots n;$$

therefore, the number of appropriate values for b is

$$= a - 2 + n - a - 3 = n - 5,$$

provided that a is neither 1 nor $a + 2 = n$. We put aside these two cases; and the number of the others being $= n - 4$, of which each is able to exist in $n - 5$ various different ways, the number of cases is

$$= (n-4)(n-5).$$

But the first sequence 1, 2, 3, is able to be combined with $n - 4$ different numbers b, and also the same for the last, $n-2,\ n-1,\ n$; whence the number of all the cases for this species is

$$= (n-4)(n-5) + 2(n-4) = (n-3)(n-4),$$

of which all are different, and thus the probability that this species exists is

$$= 2 \cdot \frac{3 \cdot 4(n-3)(n-4)}{n(n-1)(n-2)(n-3)} = 2 \cdot \frac{3 \cdot 4(n-4)}{n(n-1)(n-2)}.$$

III. For the third species, $a,\ a+1,\ b,\ b+1$, the first sequence, $a,\ a+1$, being taken at will, this which is able to be made in $n-1$ different ways [§ 2], the second sequence, $b,\ b+1$, must be taken either from this progression

$$1,\ 2,\ 3,\ \ldots a-2,$$

this which is able to occur in $a-3$ ways, or from this

$$a+3,\ a+4,\ a+5,\ \ldots n,$$

this which is able to occur in $n-a-3$ ways, provided that the number a is not 1 or 2, and $a+1$ neither n nor $n-1$; we put to the side these 4 cases; and the number of the others being $n-5$, of which each is able to occur in $n-6$ ways, the number of cases will be $=(n-5)(n-6)$. But the first sequence 1, 2, is able to be combined with $n-4$ other similar sequences, likewise the last, $n-1,\ n$; and the second, 2, 3, with $n-5$, likewise the last but one, $n-2,\ n-1$; therefore, to the number of cases already found, it is necessary again to add

$$2(n-4)+2(n-5)=4n-18,$$

such that the entire number of cases is

$$nn-11n+30+4n-18=nn-7n+12=(n-3)(n-4).$$

But here each case is encountered twice, according as one considers in the first place either the one or the other sequence. Consequently, the correct number of cases which produce these third species is

$$=\frac{(n-3)(n-4)}{1\cdot 2}$$

and the probability that this case exists

$$=\frac{3\cdot 4(n-3)(n-4)}{n(n-1)(n-2)(n-3)}=\frac{3\cdot 4(n-4)}{n(n-1)(n-2)}.$$

IV. For the fourth species, $a,\ a+1,\ b,\ c$, the sequence $a,\ a+1$ being taken at will, this which can be made in $n-1$ different ways, the two other numbers b and c must be taken in these two progressions

$$1,\ 2,\ 3,\ \ldots a-2, \quad | \quad a+3,\ a+4, \ldots n,$$

such that they do not comprise a sequence. Therefore, taking both from the first progression, of which the number of terms is $a-2$, this is able to be made in

$$\frac{(a-3)(a-4)}{1\cdot 2}$$

different ways [§ 2]; similarly, if one takes both from the other progression, of which the number of terms is $n-a-2$, this is able to occur in as many ways as

$$\frac{(n-a-3)(n-a-4)}{1\cdot 2}.$$

Now, taking b from the first and c from the second progression, the number of cases is

$$= (a-2)(n-a-2),$$

provided that one excepts the first two and the last two sequences. Therefore the number of these where this denumeration takes place being $= n-5$, of which each is able to occur as many times as

$$\frac{(a-3)(a-4)}{1\cdot 2} + \frac{(n-a-3)(n-a-4)}{1\cdot 2} + (a-2)(n-a-2),$$

this which reduces to

$$\frac{nn-11n+32}{1\cdot 2},$$

which it is necessary to multiply by $n-5$.

Now, the first sequence, 1, 2, gives as many cases as

$$\frac{(n-4)(n-5)}{1\cdot 2},$$

and as many the last, $n-1$, n; and the second, 2, 3, gives

$$\frac{(n-5)(n-6)}{1\cdot 2}$$

cases and as many the second last; such that the number of these 4 cases is

$$= (n-4)(n-5) + (n-5)(n-6) = (n-5)(2n-10) = 2nn - 20n + 50,$$

which, being added to the preceding, gives

$$\frac{n-5}{2}(nn - 11n + 32 + 4n - 20) = \frac{(n-5)(n-4)(n-3)}{1\cdot 2}$$

for the number of all the cases which produce this species, and therefore the probability that it exists is

$$= \frac{3\cdot 4(n-4)(n-5)}{n(n-1)(n-2)}.$$

V. The fifth species has no need to be developed separately, since the number of cases of all five species must be

$$= \frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4};$$

therefore we add together the cases found for the four species, of which the sum is

$$n-3 + (n-3)((n-4) + \frac{(n-3)(n-4)}{1\cdot 2} + \frac{(n-3)(n-4)(n-5)}{1\cdot 2}$$

$$= \frac{(n-3)(nn-6n+10)}{1\cdot 2}$$

which, being subtracted from $\frac{n(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3\cdot 4}$, leaves

$$\frac{n-3}{24}(n(n-1)(n-2)-12(nn-6n+10))=\frac{(n-3)(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3\cdot 4}$$

for the number of all the cases where there is no sequence among the 4 drawn numbers; whence the probability that this species exists is

$$=\frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}.$$

COROLLARY 1

12. Therefore here are the numbers of cases which produce each of the five reported species:

	Number of cases
I. Species $a,\ a+1,\ a+2,\ a+3$	$\frac{n-3}{1}$
II. Species $a,\ a+1,\ a+2,\ b$	$\frac{2(n-3)(n-4)}{1\cdot 2}$
III. Species $a,\ a+1,\ b,\ b+1$	$\frac{(n-3)(n-4)}{1\cdot 2}$
IV. Species $a,\ a+1,\ b,\ c$	$\frac{3(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3}$
V. Species $a,\ b,\ c,\ d$	$\frac{(n-3)(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3\cdot 4}$.

I have expressed these numbers so that one can maybe form soon an induction for some more complicated questions.

COROLLARY 2

13. In the same manner I will express the probability that each of these five species exist:

	Probability
I. Species $a,\ a+1,\ a+2,\ a+3$	$\frac{2\cdot 3\cdot 4}{n(n-1)(n-2)}$
II. Species $a,\ a+1,\ a+2,\ b$	$2\cdot\frac{3\cdot 4(n-4)}{n(n-1)(n-2)}$
III. Species $a,\ a+1,\ b,\ b+1$	$\frac{3\cdot 4(n-4)}{n(n-1)(n-2)}$
IV. Species $a,\ a+1,\ b,\ c$	$3\cdot\frac{4(n-4)(n-5)}{n(n-1)(n-2)}$
V. Species $a,\ b,\ c,\ d$	$\frac{(n-4)(n-5)(n-6)}{n(n-1)(n-2)}$.

PROBLEM 4

14. *The number of tickets marked with the natural numbers 1, 2, 3, etc. being some number* $=n$, *if one draws 5 at random, to find all of the probabilities which are able to take place with regard to sequences.*

SOLUTION

Among the 5 drawn numbers it is necessary to distinguish the following species to which all the possible cases, of which the number is

$$= \frac{n(n-1)(n-2)(n-3)(n-4)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

reduce themselves.

I. Species $a,\ a+1,\ a+2,\ a+3,\ a+4$, where there is a sequence of 5.

II. Species $a,\ a+1,\ a+2,\ a+3,\ b$, where there is only a sequence of 4.

III. Species $a,\ a+1,\ a+2,\ b,\ b+1$, where there is only a sequence of three and one of two.

IV. Species $a,\ a+1,\ a+2,\ b,\ c$, where there is one of three.

V. Species $a,\ a+1,\ b,\ b+1,\ c$, where there are only two of two.

VI. Species $a,\ a+1,\ b,\ c,\ d$, where there is only one of two.

VII. Species $a,\ b,\ c,\ d,\ e$ where there is no sequence.

We examine separately each of these 7 species.[1]

I. The first only contains this case:

$$(1,2,3,4,5),\ (2,3,4,5,6) \text{ etc. up to } (n-4, n-3, n-2, n-1, n),$$

of which the number is $= n-4$, and therefore the probability [that it is a sequence of five]

$$= \frac{2 \cdot 3 \cdot 4 \cdot 5}{n(n-1)(n-2)(n-3)}.$$

II. In the second species, the sequence $a,\ a+1,\ a+2,\ a+3$ is able to vary in $n-3$ different ways; and the number b must be taken from one of the two progressions

$$1,\ 2,\ 3, \ldots a-2 \quad \text{or} \quad a+5,\ a+6,\ \ldots n,$$

the number of these values is

$$= a-2+(n-a-4) = n-6,$$

with the exception of the first and last sequence. Therefore putting these two apart, the number of the others being $n-5$, those of the cases will be

$$= (n-5)(n-6).$$

[1]This phrase of the manuscript is not found in the original edition.

Now, for the first sequence, 1, 2, 3, 4, the number of values of b is $= n-5$ and also for the last. Therefore we add still these $2(n-5)$ cases to the number found $(n-5)(n-6)$, and we will have the number of all the cases which correspond to this species

$$= (n-5)(n-4) = 2 \cdot \frac{(n-4)(n-5)}{1 \cdot 2},$$

whence one extracts the probability [that there is a sequence of four]

$$= 2 \cdot \frac{3 \cdot 4 \cdot 5(n-5)}{n(n-1)(n-2)(n-3)}.$$

III. In the third species, $a, a+1,\ a+2,\ b,\ b+1$, the first sequence of three, $a,\ a+1,\ a+2$, is able to take place in $n-2$ different ways, and the other sequence of two, $b,\ b+1$, must be taken either from this progression

$$1,\ 2,\ 3,\ \ldots a-2,$$

whence their number will be $= a-3$, or from this progression

$$a+4,\ a+5,\ \ldots n,$$

whence the number of cases becomes $= n-a-4$; and therefore, the number of values of b is $= n-7$, excepting the first two and the last two sequences of three. The number of the others being therefore $= n-2-4 = n-6$, and each receiving $n-7$ cases, the number of cases is

$$= (n-6)(n-7).$$

But the first, 1, 2, 3, admits $n-5$ cases and the second $n-6$; whence the first two and the last two provide again

$$2(n-5) + 2(n-6) = 4n-22$$

cases, which being added to $(n-6)(n-7)$ produces the number of all the cases of this species

$$= nn - 9n + 20 = (n-4)(n-5) = 2 \cdot \frac{(n-4)(n-5)}{1 \cdot 2}$$

as in the preceding; whence the probability [that there is a sequence of three and one of two] is also

$$= 2 \cdot \frac{3 \cdot 4 \cdot 5(n-5)}{n(n-1)(n-2)(n-3)}.$$

IV. In the fourth species, $a,\ a+1,\ a+2,\ b,\ c$, the sequence of three takes place in $n-2$ cases; and the two numbers b and c must be taken from these two progressions

$$1,\ 2,\ 3,\ \ldots a-2, \quad | \quad a+4,\ a+5,\ \ldots n,$$

nevertheless such that they not make a sequence. We take first both in the first progression, and the number of cases is

$$\frac{(a-3)(a-4)}{1 \cdot 2};$$

but if we take them from the other, the number of cases is

$$= \frac{(n-a-3)9n-a-4)}{1\cdot 2};$$

finally, taking one from the one and the other from the other, the number of cases is

$$= (a-2)(n-a-3).$$

Thus, for each sequence of three, we have as many cases

$$\frac{(a-3)(a-4)}{1\cdot 2} + \frac{(n-a-4)(n-a-5)}{1\cdot 2} + (a-2)(n-a-3) = \frac{nn-13n+44}{1\cdot 2},$$

excepting the first two and the last two sequences; therefore, the number of those where this denumeration is correct being $= n-2-4 = n-6$, the number of cases which correspond to them is

$$\frac{(n-6)(nn-13n+44)}{2}.$$

Now, the first and the last sequence each give

$$\frac{(n-5)(n-6)}{2}$$

cases, and the second and the next to last each

$$\frac{(n-6)(n-7)}{2};$$

therefore, to the number of cases found it is necessary again to add $(n-6)(2n-12)$, whence results the sum

$$\frac{(n-6)(nn-9n+20)}{2} = \frac{(n-4)(n-5)(n-6)}{1\cdot 2},$$

which expresses the number of cases for this species; and therefore the probability [that there is a single sequence of three] is

$$3\cdot\frac{4\cdot 5\cdot (n-5)(n-6)}{n(n-1)(n-2)(n-3)}.$$

V. For the fifth species, $a,\ a+1,\ b,\ b+1,\ c$, we consider the number c; and the two sequences of two must be taken from these two progressions

$$1,\ 2,\ 3,\ \ldots c-2, \quad | \quad c+2,\ c+3,\ \ldots n.$$

If one takes both from the first, the number of cases [§ 11, III] is

$$= \frac{(c-5)(c-6)}{2},$$

and if one takes from the other, it is

$$= \frac{(n-c-4)(n-c-5)}{2}.$$

But, the one being taken from the first and the other from the last, the number of cases will be

$$= (c-3)(n-c-2);$$

therefore, for each number c, the number of cases will be

$$\frac{(c-5)(c-6)}{2} + \frac{(n-c-4)(n-c-5)}{2} + (c-3)(n-c-2) = \frac{nn-15n+62}{2}.$$

But it is necessary to exclude the first four and the last four numbers c, so that this denumeration only takes place for $n-8$ values of c, to which correspond this number of cases

$$\frac{(n-8)(nn-15n+62)}{2}.$$

Let now the other values which admit the following cases:

	Number of cases
if $c=1$ or $c=n$,	$\frac{(n-5)(n-6)}{2}$,
if $c=2$ or $c=n-1$,	$\frac{(n-6)(n-7)}{2}$,
if $c=3$ or $c=n-2$,	$\frac{(n-7)(n-8)}{2}$,
if $c=4$ or $c=n-3$,	$\frac{(n-8)(n-9)}{2}+n-6$,

of which the sum is $= 2nn - 27n + 94$, and of which the double, $4nn - 54n + 188$, must be added to the preceding number

$$\frac{n^3 - 23nn + 182n - 496}{2}$$

in order to have the number of cases

$$\frac{n^3 - 15nn + 74n - 120}{2} = \frac{(n-4)(n-5)(n-6)}{1 \cdot 2},$$

and therefore the probability [that there are two sequences of two]

$$= 3 \cdot \frac{4 \cdot 5 \cdot (n-5)(n-6)}{n(n-1)(n-2)(n-3)},$$

which is precisely the same as that of the preceding species.

VI. For the sixth species, $a,\ a+1,\ b,\ c,\ d,$ I will trace another route in considering the least of the numbers $a,\ b,\ c,\ d$. Therefore let firstly a be the least; and the three singletons $b,\ c,\ d$ will be taken in this progression

$$a+3,\ a+4,\ \ldots n,$$

of which the number of terms is $= n - a - 2$, and therefore the number of cases [§ 7]

$$= \frac{(n-a-4)(n-a-5)(n-a-6)}{1\cdot 2\cdot 3},$$

whence we will have for each number a the following cases:

if a is	the number of cases will be	now the sum of this progression is found
1	$\frac{(n-5)(n-6)(n-7)}{1\cdot2\cdot3}$	
2	$\frac{(n-6)(n-7)(n-8)}{1\cdot2\cdot3}$	
3	$\frac{(n-7)(n-8)(n-9)}{1\cdot2\cdot3}$	$= \frac{(n-4)(n-5)(n-6)(n-7)}{1\cdot2\cdot3\cdot4}$
$\vdots$	....	
$n-7$	$\frac{3\cdot2\cdot1}{1\cdot2\cdot3}$	

Next, if one of the solitary numbers, d, is least, the four others, $a,\ a+1,\ b,\ c$, will be taken from the progression

$$d+2,\ d+3, \ldots n,$$

of which the number of terms is $= n - d - 1$. Now then, the number of cases [§ 11, IV] is

$$= \frac{3(n-d-4)(n-d-5)(n-d-6)}{1\cdot 2\cdot 3}.$$

Consequently:

if d is	the number of cases will be	now the sum of this progression is found
1	$\frac{3(n-5)(n-6)(n-7)}{1\cdot2\cdot3}$	
2	$\frac{3(n-6)(n-7)(n-8)}{1\cdot2\cdot3}$	$= \frac{3(n-4)(n-5)(n-6)(n-7)}{1\cdot2\cdot3\cdot4}$
$\vdots$	....	
$n-7$	$3\cdot\frac{3\cdot2\cdot1}{1\cdot2\cdot3}$	

Consequently, the sum of all the cases which produce this species is

$$= 4\cdot\frac{(n-4)(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3\cdot 4},$$

and therefore the probability [that there is only a single sequence of two]

$$= 4\cdot\frac{5(n-5)(n-6)(n-7)}{n(n-1)(n-2)(n-3)}.$$

VII. Finally, for the seventh species, the number of all the cases which produce it is

$$= \frac{(n-4)(n-5)(n-6)(n-7)(n-8)}{1\cdot 2\cdot 3\cdot 4\cdot 5}$$

and therefore the probability [that this is no sequence]

$$= \frac{(n-5)(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)}.$$

COROLLARY 1

15. We place before the eyes at the same time the numbers of the cases and the probabilities that we come to find for the seven species of drawings, when one draws five numbers from n.

	Number of cases	Probability
I. $a,\ a+1,\ a+2,\ a+3,\ a+4$	$\frac{n-4}{1}$	$\frac{2\cdot3\cdot4\cdot5}{n(n-1)(n-2)(n-3)}$
II. $a,\ a+1,\ a+2,\ a+3,\ b$	$2\cdot\frac{(n-4)(n-5)}{1\cdot2}$	$2\cdot\frac{3\cdot4\cdot5(n-5)}{n(n-1)(n-2)(n-3)}$
III. $a,\ a+1,\ a+2,\ b,\ b+1$	$2\cdot\frac{(n-4)(n-5)}{1\cdot2}$	$2\cdot\frac{3\cdot4\cdot5(n-5)}{n(n-1)(n-2)(n-3)}$
IV. $a,\ a+1,\ a+2,\ b,\ c$	$3\cdot\frac{(n-4)(n-5)(n-6)}{1\cdot2\cdot3}$	$3\cdot\frac{4\cdot5(n-5)(n-6)}{n(n-1)(n-2)(n-3)}$
V. $a,\ a+1,\ b,\ b+1,\ c$	$3\cdot\frac{(n-4)(n-5)(n-6)}{1\cdot2\cdot3}$	$3\cdot\frac{4\cdot5(n-5)(n-6)}{n(n-1)(n-2)(n-3)}$
VI. $a,\ a+1,\ b,\ c,\ d$	$4\cdot\frac{(n-4)(n-5)(n-6)(n-7)}{1\cdot2\cdot3\cdot4}$	$4\cdot\frac{5(n-5)(n-6)(n-7)}{n(n-1)(n-2)(n-3)}$
V. $a,\ b,\ c,\ d,\ e$	$\frac{(n-4)(n-5)(n-6)(n-7)(n-8)}{1\cdot2\cdot3\cdot4\cdot5}$	$\frac{(n-5)(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)}$

COROLLARY 2

15[a].[2] If one requires the probability that there is at least one sequence of two among the five drawn numbers, all the species, except the last, satisfy, and since the sum of all the probabilities is 1, the sought probability is

$$= 1 - \frac{(n-5)(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)}.$$

COROLLARY 3

16. If one requires the probability that there is, among the five drawn numbers, at least two sequences of two, since a sequence of 3 contains two of 2 in itself, all the species, without the last two, satisfy, and therefore the sought probability will be

$$= 1 - \frac{(n-5)(n-6)(n-7)(n+12)}{n(n-1)(n-2)(n-3)}.$$

COROLLARY 4

17. Now, the probability that there will be found among the 5 drawn numbers [at least] a sequence of three will be

$$\frac{3\cdot4\cdot5(2+4(n-5)+(n-5)(n-6))}{n(n-1)(n-2)(n-3)} = \frac{3\cdot4\cdot5(n-4)}{n(n-1)(n-2)},$$

and that there is [at least] a sequence of 4, the probability will be

$$= \frac{2\cdot3\cdot4\cdot5(n-4)}{n(n-1)(n-2)(n-3)}$$

[2]The original edition places here, by error, the number 15 for a second time.

Finally, that there is a sequence of all five, the probability is

$$= \frac{2 \cdot 3 \cdot 4 \cdot 5}{n(n-1)(n-2)(n-3)}.$$

APPLICATION TO THE GENOISE LOTTERY

18. In order to apply these formulas to the Genoise Lottery, where one draws each time 5 numbers from 90, we will have $n = 90$, and in order to mark more distinctly the different cases with respect to the sequences, I will employ the following types:

(1) marks a single number, out of the entire sequence,
(2) marks a sequence of two,
(3) marks a sequence of three,
(4) marks a sequence of four,
(5) marks a sequence of all five.

This put, we will have for each of the seven different species the following probabilities:

Specie	Probability=
I. (5)	$\frac{2\cdot3\cdot4\cdot5}{90\cdot89\cdot88\cdot87}=\frac{1}{511\,038}$
II. (4), (1)	$\frac{2\cdot3\cdot4\cdot5\cdot85}{90\cdot89\cdot88\cdot87}=\frac{85}{511\,038}$
III. (3), (2)	$\frac{2\cdot3\cdot4\cdot5\cdot85}{90\cdot89\cdot88\cdot87}=\frac{85}{511\,038}$
IV. (3), (1), (1)	$\frac{3\cdot4\cdot5\cdot85\cdot84}{90\cdot89\cdot88\cdot87}=\frac{3570}{511\,038}$
V. (2), (2), (1)	$\frac{3\cdot4\cdot5\cdot85\cdot84}{90\cdot89\cdot88\cdot87}=\frac{3570}{511\,038}$
VI. (2), (1), (1), (1)	$\frac{4\cdot5\cdot85\cdot84\cdot83}{90\cdot89\cdot88\cdot87}=\frac{98770}{511\,038}$
VII. (1), (1), (1), (1), (1)	$\frac{85\cdot84\cdot83\cdot82}{90\cdot89\cdot88\cdot87}=\frac{404\,957}{511\,038}$

From this I draw the following conclusions:

1°. That among the five drawn numbers there is found at least one sequence of two, the probability is $\frac{106\,081}{511\,038}$; therefore, if one were permitted to bet on this case, the gain must be fixed at $4\frac{817}{1000}$ times the wager,[3] and if one gives only 4 times the wager, the Lottery will gain [817 on 4817 or] 17 percent.

2°. That among the 5 drawn numbers there is found at least two sequences of 2, either one of three, or several, the probability is$= \frac{7311}{511038}$; and therefore, in permitting to bet on this case, the gain must be fixed at $69\frac{9}{10}$ times the wager; therefore, if one gives only 50 times the wager, the Lottery will gain 199 on 699 or $28\frac{1}{2}$ percent.

3°. That among the five drawn numbers there is found a sequence of three or several, the probability is $= \frac{3741}{511038}$; and therefore, in permitting to bet on this case, the gain must be fixed at $136\frac{6}{10}$ times the wager; therefore, if one gives only 90 times the wager, one will gain 466 on 1366 or else $34\frac{1}{9}$ percent.

[3] By the equality of the mathematical expectations, one obtains, in designating by m the wager, the equation $x \cdot \frac{106\,081}{511\,038} \cdot m = m$, whence the ratio of x of the gain to the wager, $x = 511038 : 106081 = 4.81743\ldots$

4° That among the five drawn numbers there is found a sequence of 4, or of all five, the probability is $= \frac{86}{511038}$; and therefore, if one bets on this case, the gain must be fixed at $5942\frac{3}{10}$ times the wager;[4] therefore, if the Lottery gives only 3000 times the wager, it will gain $2942\frac{3}{10}$ on $5942\frac{3}{10}$ or else $49\frac{1}{2}$ percent.

REMARK

19. Any difficulty that had appeared at first in extending these researches to some greater number of drawn tickets, the particular route that I have employed in the solution of this problem for species VI renders these researches very easy, so that one will be in a state of extending them likewise to as great number of drawn tickets as one will wish. This entire method reverts to finding the sum of such a descending progression

$$\frac{k(k-1)(k-2)\cdots(k-m+1)}{1\cdot 2\cdot 3\cdots m}+\frac{(k-1)(k-2)\cdots(k-m)}{1\cdot 2\cdot 3\cdots m}+\text{etc.},$$

up to the one that the terms vanish. Now, one knows that the sum of this progression is expressed very simply as

$$\frac{(k+1)k(k-1)(k-2)\cdots(k-m+1)}{1\cdot 2\cdot 3\cdot 4\cdots(m+1)}.$$

I will use therefore this method to resolve the following problem.

PROBLEM 5

20. *The number of tickets marked with the natural numbers 1, 2, 3, etc. being some number*$= n$*, if one draws six at random, to find all the probabilities which are able to take place with regard to the sequences.*

SOLUTION

It is easy to establish all the different species which are able to be encountered among the six drawn numbers, that I will develop one after the other.

I. *Species.* $a,\ a+1,\ a+2,\ a+3,\ a+4,\ a+5.$

The number of cases is here obviously $= \frac{n-5}{1}$; therefore, since the number of all possible cases is

$$= \frac{n(n-1)(n-2)(n-3)(n-4)(n-5)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$

the probability that this species takes place is

$$= \frac{2\cdot 3\cdot 4\cdot 5\cdot 6}{n(n-1)(n-2)(n-3)(n-4)}.$$

II. *Species.* $a,\ a+1,\ a+2,\ a+3,\ a+4,\ b.$

[4] The original has: ...fixed at 5940 times the wager; therefore...he will win 2940 on 5940...

Let a be the least of the drawn numbers; and b must be taken from this progression

$$a+6,\ a+7,\ a+8,\ \ldots n,$$

of which the number of terms is $= n-a-5$, which gives the number of all values of b for each number a. But if b is the least of the drawn numbers, the sequence of five, $a,\ a+1,\ a+2,\ a+3,\ a+4$, must be taken from this progression

$$b+2,\ b+3,\ \ldots n,$$

of which the number of terms is $= n-b-1$; this which, by problem 4 [§ 14, I], is able to occur in

$$\frac{n-b-5}{1}$$

different ways. Thus, to find the number of cases, we have only to sum the two following progressions:

if a is	values of b	if b is	values of a
1	$\frac{n-6}{1}$	1	$\frac{n-6}{1}$
2	$\frac{n-7}{1}$	2	$\frac{n-7}{1}$
3	$\frac{n-8}{1}$	3	$\frac{n-8}{1}$
etc.	etc.	etc.	etc.
sum=	$\frac{(n-5)(n-6)}{1\cdot 2}$	sum=	$\frac{(n-5)(n-6)}{1\cdot 2}$

Therefore, the number of cases is

$$= 2\cdot\frac{(n-5)(n-6)}{1\cdot 2},$$

and the probability

$$= 2\cdot\frac{3\cdot 4\cdot 5\cdot 6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}.$$

III. *Species.* $a,\ a+1,\ a+2,\ a+3,\ b,\ b+1.$

If a is the least of the drawn numbers, the sequence of two, $b,\ b+1$, must be taken from this progression

$$a+5,\ a+6,\ a+7,\ \ldots n,$$

of which the number of terms $= n-a-4$. Therefore, by problem 1 [§ 2], the number of values of b is

$$= \frac{n-a-5}{1}.$$

If b is the least of the 6 drawn numbers, the sequence of four, $a,\ a+1,\ a+2,\ a+3$, must be taken from this progression

$$b+3,\ b+4,\ \ldots n,$$

of which the number of terms $= n - b - 2$. Therefore, by problem 3 [§ 11, I], the number of values of a is

$$= \frac{n-b-5}{1}.$$

Therefore here are the two progressions that we have to sum:

if a is	values of b	if b is	values of a
1	$\frac{n-6}{1}$	1	$\frac{n-6}{1}$
2	$\frac{n-7}{1}$	2	$\frac{n-7}{1}$
3	$\frac{n-8}{1}$	3	$\frac{n-8}{1}$
etc.	etc.	etc.	etc.
sum=	$\frac{(n-5)(n-6)}{1\cdot 2}$	sum=	$\frac{(n-5)(n-6)}{1\cdot 2}$

Therefore, the number of cases is

$$= 2 \cdot \frac{(n-5)(n-6)}{1 \cdot 2},$$

and the probability

$$= 2 \cdot \frac{3 \cdot 4 \cdot 5 \cdot 6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}.$$

IV. *Species.* $a,\ a+1,\ a+2,\ a+3,\ b,\ c$.

Let a be the least of the drawn numbers; and the two singletons b, c, must be taken from this progression

$$a+5,\ a+6, \ldots n,$$

of which the number of terms $= n - a - 4$. Therefore, by problem 1 [§ 2], the number of cases is

$$= \frac{(n-a-5)(n-a-6)}{1 \cdot 2}.$$

Let one of the single numbers, c, be the least; and the sequence of 4 with the other solitary, $a,\ a+1,\ a+2,\ a+3,\ b$, must be taken from this progression

$$c+2,\ c+3,\ \ldots n,$$

of which the number of terms $= n - c - 1$. Therefore, by problem 4 [§ 15, II], the number of cases is

$$= 2 \cdot \frac{(n-c-5)(n-c-6)}{1 \cdot 2}.$$

We will have therefore to sum the following two progressions:

if a is	the number of cases	if c is	the number of cases
1	$\frac{(n-6)(n-7)}{1\cdot 2}$	1	$2 \cdot \frac{(n-6)(n-7)}{1\cdot 2}$
2	$\frac{(n-7)(n-8)}{1\cdot 2}$	2	$2 \cdot \frac{(n-7)(n-8)}{1\cdot 2}$
3	$\frac{(n-8)(n-9)}{1\cdot 2}$	3	$\frac{(n-8)(n-9)}{1\cdot 2}$
etc.	etc.	etc.	etc.
sum=	$\frac{(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3}$	sum=	$2 \cdot \frac{(n-5)(n-6)(n-7)}{1\cdot 2\cdot 3}$

Therefore, the number of cases is

$$= 3 \cdot \frac{(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3},$$

and the probability

$$= 3 \cdot \frac{4 \cdot 5 \cdot 6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}.$$

V. *Species.* $a,\ a+1,\ a+2,\ b,\ b+1,\ b+2.$

Since there are here two similar sequences of three, it is indifferent which of the two numbers a and b is the least; and the sequence of three, $a,\ a+1,\ a+2$, must be taken from this progression

$$b+4,\ b+5,\ \ldots n,$$

of which the number of terms is $= n-b-3$. Therefore, by problem 2 [§ 7], the number of cases is

$$= \frac{n-b-5}{1},$$

and the progression to sum

$$\frac{n-6}{1},\ \frac{n-7}{1},\ \frac{n-8}{1} \text{ etc.}$$

Therefore, the number of cases is

$$= \frac{(n-5(n-6)}{1 \cdot 2},$$

and the probability

$$= \frac{3 \cdot 4 \cdot 5 \cdot 6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}.$$

VI. *Species.* $a,\ a+1,\ a+2,\ b,\ b+1,\ c.$

If a is the least of the drawn numbers, the others, $b,\ b+1,\ c$, must be taken from this progression

$$a+4,\ a+5,\ \ldots n,$$

of which the number of terms $= n-a-3$; whence, [by] problem 2 [§ 7], the number of cases is

$$= 2 \cdot \frac{(n-a-5)(n-a-6)}{1 \cdot 2},$$

and putting $a = 1$, one has

$$2 \cdot \frac{(n-6)(n-7)}{1 \cdot 2}.$$

If b is the least of the drawn numbers, the others, $a,\ a+1,\ a+2,\ c,$ must be taken from this progression

$$b+3,\ b+4,\ \ldots n,$$

of which the number of terms is $n-b-2$; whence, by problem 3 [§ 11, II], the number of cases is

$$= 2 \cdot \frac{(n-b-5)(n-b-6)}{1 \cdot 2},$$

and putting $b = 1$, one has

$$2 \cdot \frac{(n-6)(n-7)}{1 \cdot 2}.$$

If c is the least, the others, $a,\ a+1,\ a+2,\ b,\ b+1$, must be taken from this progression

$$c+2,\ c+3, \ldots n,$$

of which the number of terms $= n-c-1$, whence, by problem 4 [§ 15, III], the number of cases is

$$= 2 \cdot \frac{(n-c-5)(n-c-6)}{1 \cdot 2},$$

and putting $c = 1$, one has

$$2 \cdot \frac{(n-6)(n-7)}{1 \cdot 2}.$$

The three progressions to sum reduce therefore to this alone

$$6 \cdot \frac{(n-6)(n-7)}{1 \cdot 2} + 6 \cdot \frac{(n-7)(n-8)}{1 \cdot 2} + 6 \cdot \frac{(n-8)(n-9)}{1 \cdot 2} + \text{etc.}$$

Therefore, the number of cases is

$$= 6 \cdot \frac{(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3},$$

and the probability

$$= 6 \cdot \frac{4 \cdot 5 \cdot 6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}.$$

VII. *Species.* $a,\ a+1,\ a+2,\ b,\ c,\ d.$

If a [is] the least of the drawn numbers, instead of which one is able at first to take unity, the others, b, c, d, must be taken from this progression

$$5, 6,\ 7, \ldots n,$$

of which the number of terms $= n-4$. Therefore, by problem 2 [§ 7], the number of cases is

$$= \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3}.$$

If one of the singletons is least, as $d = 1$, the others, $a,\ a+1,\ a+2,\ b,\ c$, must be taken from this progression

$$3,\ 4,\ 5,\ \ldots n,$$

of which the number of terms $= n - 2$. Therefore, by problem 4 [§ 15, IV], the number of cases

$$= 3 \cdot \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3}.$$

The concern therefore is to sum the descending progression which begins with

$$4 \cdot \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3}.$$

Therefore, the number of all the cases is

$$= 4 \cdot \frac{(n-5)(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3 \cdot 4},$$

and the probability

$$= 4 \cdot \frac{5 \cdot 6(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)(n-4)}.$$

VIII. *Species.* $a,\ a+1,\ b,\ b+1,\ c,\ c+1.$

Here, there is only one case to consider. Therefore let $c = 1$: and the others, $a,\ a+1,\ b,\ b+1$, must be drawn from this progression

$$4,\ 5,\ 6,\ \ldots n,$$

of which the number of terms $= n - 3$, whence by problem 3 [§ 12, III], the number of cases

$$= \frac{(n-6)(n-7)}{1 \cdot 2},$$

which gives the progression to sum. Therefore, the number of all the cases is

$$= \frac{(n-5)(n-6)(n-7)}{1 \cdot 2 \cdot 3},$$

and the probability

$$= \frac{4 \cdot 5 \cdot 6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}.$$

IX. *Species.* $a,\ a+1,\ b,\ b+1,\ c,\ d.$

Let $a = 1$; and the others, $b,\ b+1,\ c,\ d$, must be drawn from this progression

$$4,\ 5,\ 6,\ \ldots n,$$

of which the number of terms $= n - 3$. Therefore, by problem 3 [§ 12, IV], the number of cases

$$= 3 \cdot \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3},$$

and in this formula is already contained the position $b = 1$.

Let $d = 1$, where the letter c is already included; the others, a, $a+1$, b, $b+1$, c, must be drawn from this progression

$$3,\ 4,\ 5,\ \ldots n,$$

of which the number of terms $= n-2$. Therefore, by problem 4 [§ 15, V], the number of cases

$$= 3 \cdot \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3},$$

and therefore one has only to sum the descending progression which begins with the term

$$= 6 \cdot \frac{(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3}.$$

Whence the number of all the cases

$$= 6 \cdot \frac{(n-5)(n-6)(n-7)(n-8)}{1 \cdot 2 \cdot 3 \cdot 4},$$

and the probability

$$= 6 \cdot \frac{5 \cdot 6(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)(n-4)}.$$

X. *Species.* $a,\ a+1,\ b,\ c,\ d,\ e.$

Let firstly $a = 1$; and the four singleton numbers b, c, d, e must be drawn from this progression

$$4,\ 5,\ 6,\ \ldots n,$$

of which the number of terms $= n-3$; and, by problem 3 [§ 12, V], the number of cases

$$= \frac{(n-6)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

Let next one of the singletons, e,$= 1$; and the others, a, $a+1$, b, c, d, must be drawn from this progression

$$3,\ 4,\ 5,\ \ldots n,$$

of which the number of terms $= n-2$; and, by problem 4 [§ 15, VI], the number of cases

$$= 4 \cdot \frac{(n-6)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4},$$

such that the concern is to sum the descending progression which begins with the term

$$5 \cdot \frac{(n-6)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4}.$$

Therefore, the number of all the cases

$$= 5 \cdot \frac{(n-5)(n-6)(n-7)(n-8)(n-9)}{1 \cdot 2 \cdot 3 \cdot 4 \cdot 5},$$

and the probability

$$= 5 \cdot \frac{6(n-6)(n-7)(n-8)(n-9)}{n(n-1)(n-2)(n-3)(n-4)}.$$

XI. *Species.* a, b, c, d, e, f.

Here, there is only one case to consider, we put $f = 1$; and the others, a, b, c, d, e, must be drawn from this progression

$$3,\ 4,\ 5,\ \ldots n,$$

of which the number of terms $= n-2$. Thus, by problem 4 [§ 15, VII], the number of cases

$$= \frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{1\cdot 2\cdot 3\cdot 4\cdot 5}.$$

Therefore, the number of all the cases

$$= \frac{(n-5)(n-6)(n-7)(n-8)(n-9)(n-10)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6},$$

and the probability

$$= \frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{n(n-1)(n-2)(n-3)(n-4)}.$$

COROLLARY 1

21. In order to put all this more clearly before the eyes, I will make use of the same characters to mark the different species of sequences as I have exhibited § 18, and we will have for each species:

Species	Number of cases	Probability
I. (6)	$\frac{n-5}{1}$	$\frac{2\cdot3\cdot4\cdot5\cdot6}{n(n-1)(n-2)(n-3)(n-4)}$
II. (5)+(1)	$2\cdot\frac{(n-5)(n-6)}{1\cdot2}$	$2\cdot\frac{3\cdot4\cdot5\cdot6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}$
III. (4)+(2)	$2\cdot\frac{(n-5)(n-6)}{1\cdot2}$	$2\cdot\frac{3\cdot4\cdot5\cdot6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}$
IV. (4)+2(1)	$3\cdot\frac{(n-5)(n-6)(n-7)}{1\cdot2\cdot3}$	$3\cdot\frac{4\cdot5\cdot6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}$
V. 2(3)	$\frac{(n-5)(n-6)}{1\cdot2}$	$\frac{3\cdot4\cdot5\cdot6(n-6)}{n(n-1)(n-2)(n-3)(n-4)}$
VI. (3)+(2)+(1)	$6\cdot\frac{(n-5)(n-6)(n-7)}{1\cdot2\cdot3}$	$6\cdot\frac{4\cdot5\cdot6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}$
VII. (3)+3(1)	$4\cdot\frac{(n-5)(n-6)(n-7)(n-8)}{1\cdot2\cdot3\cdot4}$	$4\cdot\frac{5\cdot6(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)(n-4)}$
VIII. 3(2)	$\frac{(n-5)(n-6)(n-7)}{1\cdot2\cdot3}$	$\frac{4\cdot5\cdot6(n-6)(n-7)}{n(n-1)(n-2)(n-3)(n-4)}$
IX. 2(2)+2(1)	$6\cdot\frac{(n-5)(n-6)(n-7)(n-8)}{1\cdot2\cdot3\cdot4}$	$6\cdot\frac{5\cdot6(n-6)(n-7)(n-8)}{n(n-1)(n-2)(n-3)(n-4)}$
X. (2)+4(1)	$5\cdot\frac{(n-5)(n-6)(n-7)(n-8)(n-9)}{1\cdot2\cdot3\cdot4\cdot5}$	$5\cdot\frac{6(n-6)(n-7)(n-8)(n-9)}{n(n-1)(n-2)(n-3)(n-4)}$
XI. 6(1)	$\frac{(n-5)(n-6)(n-7)(n-8)(n-9)(n-10)}{1\cdot2\cdot3\cdot4\cdot5\cdot6}$	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{n(n-1)(n-2)(n-3)(n-4)}$.

COROLLARY 2

22. The law of these expressions for the number of the cases of each species is quite evident, since the number of factors in the numerators is the same as those of the different letters by which I have characterized before the different species, in commencing with $n-5$ and diminishing them by one unit; now, the denominator contains always as many factors, in commencing by 1, 2, 3, etc. But the law of the numeric coefficients is not so evident; however, it will become enough in representing the numbers of cases in the following manner.

Species	Number of cases
1(6)	$\frac{n-5}{1}$
1(5)+1(1)	$\frac{n-5\cdot n-6}{1\cdot 1}$
1(4)+1(2)	$\frac{n-5\cdot n-6}{1\cdot 1}$
1(4)+2(1)	$\frac{n-5\cdot n-6\cdot n-7}{1\cdot 1\cdot 2}$
2(3)	$\frac{n-5\cdot n-6}{1\cdot 1}$
1(3)+1(2)+1(1)	$\frac{n-5\cdot n-6\cdot n-7}{1\cdot 1\cdot 1}$
1(3)+3(1)	$\frac{n-5\cdot n-6\cdot n-7\cdot n-8}{1\cdot 1\cdot 2\cdot 3}$
3(2)	$\frac{n-5\cdot n-6\cdot n-7}{1\cdot 2\cdot 3}$
2(2)+2(1)	$\frac{n-5\cdot n-6\cdot n-7\cdot n-8}{1\cdot 2\cdot 1\cdot 2}$
1(2)+4(1)	$\frac{n-5\cdot n-6\cdot n-7\cdot n-8\cdot n-9}{1\cdot 1\cdot 2\cdot 3\cdot 4}$
6(1)	$\frac{n-5\cdot n-6\cdot n-7\cdot n-8\cdot n-9\cdot n-10}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6}$

where the denominators manifestly follow the coefficients of the sequences of each order, which characterize each species.

REMARK

23. Therefore here we are now in a position to render these researches altogether general; however, one cannot begin with the general problem, since each number of drawn tickets depends upon all the preceding. But, although this last method that I come to use is of such great advantage, nonetheless it is to be presumed that one will discover one more simple.

GENERAL PROBLEM

24. *The number of tickets marked with the natural numbers 1, 2, 3, 4 etc. being some number= n, if one draws m tickets at random, to determine all the probabilities which are able to take place with regard to sequences.*

SOLUTION

In order to distinguish the different kinds of sequences which are able to be found in each drawing of m numbers, I will use these signs [§ 18]:

(1) marks a solitary number,
(2) marks a sequence of two numbers,
(3) marks a sequence of three numbers,
(4) marks a sequence of four numbers
etc.

This put, each drawing will be characterized by such a formula

$$\alpha(a) + \beta(b) + \gamma(c) + \delta(d) + \text{etc.},$$

this which signifies that there are α sequences of a numbers, β sequences of b numbers, γ sequences of c numbers, δ sequences of d numbers etc.; and since the number of drawn numbers $= m$, it is necessary that there be

$$\alpha a + \beta b + \gamma c + \delta d + \text{etc.} = m.$$

Now we put moreover

$$\alpha + \beta + \gamma + \delta + \text{etc.} = k;$$

and the number of all the cases which produce the said drawing

$$\alpha(a) + \beta(b) + \gamma(c) + \delta(d) + \text{etc.},$$

will be expressed so

$$\frac{(n-m+1)(n-m)(n-m-1)\cdots(n-m-k+2)}{1\cdot 2\cdots\alpha\cdot 1\cdot 2\cdots\beta\cdot 1\cdot 2\cdots\gamma\cdot 1\cdot 2\cdots\delta\cdot \text{etc.}}$$

which number being divided by the number of all the possible cases, which is

$$= \frac{n(n-1)(n-2)\cdots(n-m+1)}{1\cdot 2\cdot 3\cdots m},$$

will give the probability that this same case exists. This is in what consists the complete solution of our problem.

COROLLARY 1

25. One will have therefore as many species of drawings as it is possible to find the different formulas

$$\alpha(a) + \beta(b) + \gamma(c) + \delta(d) + \text{etc.},$$

of which the sum $\alpha a + \beta b + \gamma c + \delta d$+etc. let it $= m$, that is to say as many as it is possible to partition the number m into parts in different ways; thus, taking for m successively the numbers 1, 2, 3, 4 etc., the numbers of species will form the following progression

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	
1,	2,	3,	5,	7,	11,	15,	22,	30,	42,	56,	77,	101,	135,	176	etc.

of which I have explained the nature in my researches on the partition of numbers.[5]

[5]This concerns the memoirs 158, 191 and 394 (following the index of Eneström): "Observationes analyticae variae de combinationibus," *Comment. acad. sc. Petrop.* 13 (1741/3), 1751, p. 64, "De partitione numerorum," *Novi comment. acad. sc. Petrop.* 3 (1750/1), 1753, p. 125, "De partitione numerorum in partes tam numero quam specie datas," *Novi comment. acad. sc. Petrop.* 14 (1769): I, 1770, p. 168; *LEONHARDI EULER I Opera Omnia*, series I, vol. 2, p. 163 et 254, vol. 3, p. 131 (see also the prefaces to these two volumes written by the editor M. F. Rudio). Moreover, Euler has dedicated to the partition of numbers chapter XVI of the first volume of his work *Introductio in analysin infinitorum*, Laussanne 1748, p. 253–275; *Leonhardi Euleri Opera Omnia*, series I, vol. 8, p. 313–338.

COROLLARY 2

26. The number of factors which compose the number of cases for each species $\alpha(a) + \beta(b) + \gamma(c) + \delta(d)$+etc. is always equal to the number

$$k = \alpha + \beta + \gamma + \delta + \text{etc.},$$

which marks into how many parts the number m is partitioned. And taking all the species together where k has the same value, the number of cases are jointly this sum

$$\frac{(m-1)(m-2)\cdots(m-k-1)}{1\cdot 2\cdots(k-1)} \cdot \frac{(n-m+1)(n-m)(n-m-1)\cdots(n-m-k+2)}{1\cdot 2\cdot 3\cdots k}$$

Therefore if k is	the number of cases of all species which belong to this value of k are
1	$\frac{n-m+1}{1}$,
2	$\frac{m-1}{1}\cdot\frac{(n-m+1)(n-m)}{1\cdot 2}$,
3	$\frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{(n-m+1)(n-m)(n-m-1)}{1\cdot 2\cdot 3}$,
4	$\frac{m-1}{1}\cdot\frac{m-2}{2}\cdot\frac{m-3}{3}\cdot\frac{(n-m+1)(n-m)(n-m-1)(n-m-2)}{1\cdot 2\cdot 3\cdot 4}$
etc.	etc.

and all these numbers of cases added together must produce the number of all the possible cases, which is

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-m+1)}{1\cdot 2\cdot 3\cdot 4\cdots m}.$$

COROLLARY 3

27. The first species being $1(m)$, where all m drawn numbers form one sequence, one will have $k = 1$, and the number of cases is

$$\frac{n-m+1}{1};$$

but the last species being $m(1)$, where there is no sequence, there must be $k = m$, and the number of cases

$$\frac{(n-m+1)(n-m)(n-m-1)\cdots(n-2m+2)}{1\cdot 2\cdot 3\cdots m}$$

Therefore, the probability that there is no sequence among the m drawn numbers is

$$\frac{(n-m+1)(n-m)(n-m-1)\cdots(n-2m+2)}{n(n-1)(n-2)\cdots(n-m+1)},$$

or else

$$\frac{(n-m)(n-m-1)(n-m-2)\cdots(n-2m+2)}{n(n-1)(n-2)\cdots(n-m+2)},$$

Now, this same formula expresses also the probability that of $m-1$ given numbers none are found among the m drawn numbers.

EXAMPLE

28. We make application where of n tickets marked with the numbers 1, 2, 3, ... n one draws 7 tickets; and one will find the number of cases which produce each of the 15 species which are able to take place in a drawing of seven numbers.

Species	Number of cases.
I. (7)	$\frac{n-6}{1}$,
II. (6)+1	$\frac{(n-6)(n-7)}{1\cdot 1}$,
III. (5)+2	$\frac{(n-6)(n-7)}{1\cdot 1}$,
IV. (4)+3	$\frac{(n-6)(n-7)}{1\cdot 1}$,
V. (5)+2(1)	$\frac{(n-6)(n-7)(n-8)}{1\cdot 1\cdot 2}$,
VI. (4)+(2)+(1)	$\frac{(n-6)(n-7)(n-8)}{1\cdot 1\cdot 1}$,
VII. (2(3)+(1)	$\frac{(n-6)(n-7)(n-8)}{1\cdot 2\cdot 1}$,
VIII. (3)+2(2)	$\frac{(n-6)(n-7)(n-8)}{1\cdot 1\cdot 2}$,
IX. (4)+3(1)	$\frac{(n-6)(n-7)(n-8)(n-9)}{1\cdot 1\cdot 2\cdot 3}$,
X. (3)+(2)+2(1)	$\frac{(n-6)(n-7)(n-8)(n-9)}{1\cdot 1\cdot 1\cdot 2}$,
XI. 3(2)+(1)	$\frac{(n-6)(n-7)(n-8)(n-9)}{1\cdot 2\cdot 3\cdot 1}$,
XII. (3)+4(1)	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{1\cdot 1\cdot 2\cdot 3\cdot 4}$,
XIII. 2(2)+3(1)	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)}{1\cdot 2\cdot 1\cdot 2\cdot 3}$,
XIV. (2)+5(1)	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)(n-11)}{1\cdot 1\cdot 2\cdot 3\cdot 4\cdot 5}$,
XV. 7(1)	$\frac{(n-6)(n-7)(n-8)(n-9)(n-10)(n-11)(n-12)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$.

Each of these numbers divided by the number of all the possible cases, which is

$$\frac{n(n-1)(n-2)(n-3)(n-4)(n-5)(n-6)}{1\cdot 2\cdot 3\cdot 4\cdot 5\cdot 6\cdot 7}$$

gives the probability that the corresponding species exists.