

Frederick to Euler
Potsdam, 15 September 1749

His Majesty the King of Prussia, Our very gracious Sovereign sends enclosed to professor Euler the proposal of a lottery established in most of the large cities of Italy and which had been presented to His Majesty by a certain Roccolini in order to examine with exactitude the algebraic calculations which enter into all the pieces of the proposal; but especially by good thorough study by algebra all the hazards that the entrepreneur of a similar lottery could incur and similarly the profits that he is able to make that way and making as soon as he will be able to make it his very humble report to His Majesty.

Frederick

P.S. In case that the said professor Euler have need again of some light touching this proposal, he will be able to address for this effect Lieutenant-general Count de Mottembourg.

Euler to Frederick II
Berlin, 17 September 1749

Sire,

My researches on the hydraulic machine occupying me again for some days, I take the liberty to render account of the *examination of the Italian Lottery*, for which it has pleased Your Majesty to charge me so graciously. First I have determined by the calculus of probabilities, how much each player ought to pay in order that the advantage were so much equal for him as for the bank. Thence it is next easy to conclude, how much the bank ought to earn probably, if the player is obliged to pay more than the fairness of the game demands.

Following the proposal, there are 90 tickets marked with the numbers 1, 2, 3, etc. up to 90, from which one draws 5 tickets at random, when sufficiently many players will be committed. Now one is able to participate in the game in many different ways, and each person is the master of his choosing that which agrees with him more, and of his determining even the amount of prize to which he aspires.

The 1st way is: The player chooses at will a number of the proposed 90, and he sets for himself likewise the gain, which he ought to receive in case that his number is found on one of the 5 tickets, which will be extracted when the lottery is drawn. As there are only 5 cases in 90, that his number is found among the extracted, the probability of gain for him will be $\frac{5}{90}$ or $\frac{1}{18}$: thus if the game ought to be equal on the one side and on the other, in order to win for example 100 ecus¹, he ought to pay only the 18th part in order to obtain this condition, he will pay only 5 ecus, 13 gr. 4 d. Now according to the

¹ The ecu is any of the French gold or silver coins. In Euler's time, the name referred to the silver crown. The denier was a copper coin of little value. The ecu contained 288 denier. A gros contained 12 deniers; and ecu contained 24 gros.

proposal, it is necessary that he pay 8 ecus; from which the bank ought to expect a profit of 44%. It will be the same with all the other prizes, that the players require for their manner of play: so if some one wished to win 1000 ecus, he ought to pay 80 ecus, and if his chosen number is found in the 5 extracted tickets, he would receive the sum of 1000 ecus, instead he loses his wager, if his number is not found there.

The 2nd way to play is by the *ambes* in a way: The player chooses two numbers at the time and determines also for himself the prize that the bank ought to pay him, in the case that all of the two numbers are found among the five extracted numbers. Now the probability that this occur being only $\frac{5 \cdot 4}{90 \cdot 80}$ or $\frac{2}{801}$; if the advantage ought to be equal, in order to win for example 100 ecus, he pays only $\frac{2}{801}$ part of 100 ecus, that which is nearly 6 gros. Now, according to the proposal, in order to win in this fashion 100 ecus, the wager is of 14 gros and therefore the bank would earn $133\frac{5}{8}\%$. But as this profit would be quite considerable, in order to more encourage the players, the project grants them an increase of 20% on each prize, which they will win for an ambo, and thence the profit of the bank will be reduced to $94\frac{11}{16}\%$.

The 3rd way to participate in this lottery is by *terns*; where the player chooses 3 numbers at the time, under condition of winning the prize, which he sets for himself even in case that all of these three numbers are found among the extracted 5: In this case the probability of winning for the player being $\frac{5 \cdot 4 \cdot 3}{90 \cdot 89 \cdot 88}$ or $\frac{1}{11748}$; in order to win a prize for example of 100 ecus, he ought to pay according to the law of equality only the 11748th part of 100 ecus, that which is $2\frac{1}{2}$ deniers. Now the proposal requires for this condition a payment of 15 deniers, and therefore the bank will earn $511\frac{7}{8}\%$. But for a greater encouragement the bank pledges to credit 80% on each prize won by the terns; thence the profit of the bank is reduced to 240%.

These are the three principal ways to play, but there are more of other mixtures of these. As for example a player chooses three numbers on this condition that, if all three are found among the 5 extracted numbers, he takes a prize of 50 ecus, but if he encounters there only two of the numbers, or an ambo, he draws from the bank 5 ecus. In order to determine the wager according to the law of equality one finds by the rules of combinations that for a tern it is necessary to pay $\frac{1}{11748}$ of the prize, and for the ambo $\frac{85}{11748}$ of the prize. Thus the former being supposed 50 ecus and the latter 5 ecus, the player will be obliged to pay in all a little less than a gros. I do not find how much he ought to pay in this case according to the proposal; but it is at present very easy to determine it in a way, that the bank earns as much percentage as one wishes. So if the bank would wish to earn 100%, one would make the player pay two gros instead of one.

It will be equally easy to determine, so much for these manners of play as for all the others, that one is able to imagine, how much the player ought to pay for the conditions that he demands, in order that the bank draw such profit as one wishes.

According to the proposal the profit of the bank thus will be considerable, seeing that on the simple extracts it will earn 44%, on the ambes 94% and on the terns 240%. It seems also reasonable that the bank be satisfied with a lesser profit on the simple numbers, than on the ambes and terns. For it is only in these latter cases that the bank risks losing great sums in proportion to more moderate wagers. Again such a not very

enormous profit does not discourage the player noticeably, who in order to win 100 ecus will not have great difficulty paying 15 deniers instead of $2\frac{1}{2}$ for a tern.

But for this profit, that the calculations have just shown, it is necessary to remark, that it is only probable and that it is nothing less than unassured. There could happen some throws so fortunate for the players, that the bank would be obliged to disburse well beyond the probable. Certainly one may assure in general, that the more the number of players is great, the more also the real profit will approach the calculated. This lottery is thus of such nature that it would not be to the point to interfere in it, unless one were not assured, that a very great number of players would be found. For if the number of players were small and therefore the sum of the money, which came to the bank, not very considerable, one would risk considerable loss, especially, if some players put their money for some great sums on the terns. It seems therefore that the bank must not give to some prizes too much, nor permit that many persons choose the same number: since in case, that they win, the treasury would suffer at the time a very great loss.

For the remainder one would not be bound likewise neither to the number of 90 tickets, nor to that of 5, which one draws from them. One is able to vary them to infinity, and by the rule of combinations it is easy to make always the arrangement in a way, that the bank probably earns as much for 100, as one would wish. For in seeing an example, I have imagined a proposal similar to the one, that I come to examine. I set the number of tickets to 100 in a way that all the numbers from 1 to 100 are found; I would make next to draw 10, instead of 5 according to the Italian proposal; and in order that the advantage be equal of the side of the bank and the players, the conditions of the game would be as follows:

I. The player takes only one number, in order that he win 1000 ecus in case that his number is found among the extracted 10, he ought to pay to the bank 100 ecus

II. If the player chooses two numbers, one will make two cases:
1) in order to win 1000 ecus in case that all of his two numbers would be extracted, he will pay 9 ecus, 2 gros
2) in order to win 1000 ecus when one alone of his numbers is extracted, he will pay 181 ecus, 20 gros

III. If the player chooses three 3 numbers at the time, one will have three cases to consider:
1) in order to win 1000 ecus, in case that all of his 3 numbers would be among the extracted, he will pay 18 gros
2) in order to win 1000 ecus, when there are only 2 of his numbers extracted, he will pay 25 ecus, 1 gros
3) in order to win 1000 ecus, provided that one alone of his numbers would be among the extracted, he pays 247 ecus, 1 gros

IV. If the player chooses 4 numbers, one will have these 4 cases:
1) in order to win 1000 ecus, when all of his 4 numbers would be extracted, he will pay 1 gr, 4 d.

- 2) in order to win 1000 ecus, when three of his numbers would be in the extracted, he pays 2 ecus, 21 gr.
- 3) in order to win 1000 ecus, when there are found only two of his numbers among the extracted, he pays 47 ecus, 10 gr.
- 4) in order to win 1000 ecus, when he will have only one of his numbers, he will pay 312 ecus, 17 gr.

I have supposed here for all the same gain of 1000 ecus, but these same results would be applicable to all the other gains, that the players will be able to prescribe, for so much as the prize, as one demands, will be greater or smaller than 1000 ecus, one will increase or decrease in proportion the payment, that it is necessary to make to the bank.

Next the wagers are here marked according to the equality of the advantage on the one side to the other, but one would have only to determine how much percentage the bank ought to earn for each manner of play, in order to adjust and raise the wagers accordingly. So if one would wish to earn 100%, one will only double the marked wagers.

These are the calculations and the remarks, to which the examination of the Italian proposal has lead me, and of which I wish, that they satisfy to the high intention, that Your Majesty had in mind in charging me with this Commission.

Finally all penetrated by gracious honor, of which it pleased Your Majesty to deign on me, I demand, Sire, permission to dare to present my very humble and very respectful thanks, being with the most profound respect.

Sire
of Your Majesty
the very humble, very obedient and
very dutiful subject
L. Euler