Jean Bernoulli's 5th and 6th Problems

restart:

First step, define base probability distributions. Now \( d_1 \) through \( d_6 \) are the probability distributions of the sum of the faces obtained through the cast of 1 to 6 dice. Each sequence starts with the outcome 1 and extends as far as 36.

\[
\begin{align*}
\text{d}_1 & := [\frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \frac{1}{6}, \\
& \quad \frac{1}{6}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\
\text{d}_2 & := [0, \frac{1}{36}, \frac{1}{18}, \frac{1}{12}, \frac{1}{9}, \frac{5}{36}, \frac{1}{6}, \frac{5}{36}, \frac{1}{9}, \frac{1}{12}, \frac{1}{18}, \\
& \quad \frac{1}{36}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\
\text{d}_3 & := [0, 0, 1\frac{1}{216}, 1\frac{1}{72}, 1\frac{1}{36}, 5\frac{1}{108}, 7\frac{1}{72}, 25\frac{1}{216}, 1\frac{1}{8}, \\
& \quad 1\frac{1}{8}, 25\frac{1}{216}, 7\frac{1}{72}, 5\frac{1}{72}, 5\frac{1}{108}, 1\frac{1}{36}, 1\frac{1}{72}, \\
& \quad 1\frac{1}{216}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\
\text{d}_4 & := [0, 0, 0, 1\frac{1}{1296}, 1\frac{1}{324}, 5\frac{1}{648}, 5\frac{1}{324}, 35\frac{1}{1296}, 7\frac{1}{162}, 5\frac{1}{81}, \\
& \quad 13\frac{1}{162}, 125\frac{1}{1296}, 35\frac{1}{324}, 73\frac{1}{648}, 35\frac{1}{324}, 125\frac{1}{1296}, 13\frac{1}{162}, 5\frac{1}{81}, \\
& \quad 7\frac{1}{162}, 35\frac{1}{1296}, 5\frac{1}{324}, 5\frac{1}{648}, 1\frac{1}{324}, \\
& \quad 1\frac{1}{1296}, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]; \\
\text{d}_5 & := [0, 0, 0, 0, 1\frac{1}{7776}, 5\frac{1}{7776}, 5\frac{1}{2592}, 35\frac{1}{7776}, 35\frac{1}{3888}, \\
& \quad 7\frac{432}, 205\frac{1}{7776}, 305\frac{1}{7776}, 35\frac{1}{648}, 5\frac{1}{72}, 217\frac{1}{2592}, 245\frac{1}{2592}, \\
& \quad 65\frac{1}{648}, 65\frac{1}{648}, 245\frac{1}{2592}, 217\frac{1}{2592}, 5\frac{1}{72}, 35\frac{1}{648}, 305\frac{1}{7776}, \\
& \quad 205\frac{1}{7776}, 7\frac{432}, 35\frac{1}{3888}, 35\frac{1}{7776}, 5\frac{1}{2592}, 5\frac{1}{7776}, \\
& \quad 1\frac{1}{7776}, 0, 0, 0, 0, 0, 0]; \\
\text{d}_6 & := [0, 0, 0, 0, 0, 1\frac{1}{46656}, 1\frac{1}{7776}, 7\frac{1}{15552}, 7\frac{5832}, 7\frac{2592}, \\
& \quad 7\frac{1296}, 19\frac{1}{1944}, 7\frac{432}, 43\frac{1}{1728}, 833\frac{2}{3328}, 749\frac{1}{15552}, 119\frac{1}{1944}, \\
& \quad 3431\frac{1}{46656}, 217\frac{1}{2592}, 469\frac{1}{5184}, 361\frac{1}{3888}, 469\frac{1}{5184}, 217\frac{1}{2592}, \\
& \quad 3431\frac{1}{46656}, 119\frac{1}{1944}, 749\frac{1}{15552}, 833\frac{2}{3328}, 43\frac{1}{1728}, 7\frac{432}, \\
& \quad 19\frac{1}{1944}, 7\frac{1296}, 7\frac{2592}, 7\frac{5832}, 7\frac{15552}, 1\frac{1}{7776}, 1\frac{1}{46656}];
\end{align*}
\]

Second. Analysis of the game.

The first roll of Player A determines the number of rolls of Player B. Hence Player B may roll equivalently 1 to 6 dice each with probability 1/6 and the distributions of the outcomes of Player B are as \( d_1 \) through \( d_6 \) respectively. On the other hand, Player A, given the outcome of the first roll, always has a sum distributed as \( d_2 \).

Therefore, the more direct solution of the problem must lie through conditioning on the 1st roll of A.

Third part. What is the probability of a tie? We condition on the outcome of the first die.

For example, given 1st roll yields 1, Player A can obtain any value from 2+1 to 12+1 but B can only obtain values from 1 to 6. A tie occurs if B rolls \( k \) and A rolls \( k-1 \) on the remaining two tosses. Therefore, the probability of a tie is the sum of \( d_1(k) \cdot d_2(k - 1) \) for \( k = 3 \) to 6.
Another example: Given the 1st roll yields 3, Player A can obtain any value from 2+3 to 12+3 and B any value from 3 to 18. Given B has $k$, a tie occurs if Player A rolls $k − 3$ on the remaining two tosses. The probability of a tie is therefore $d3(k) \cdot d2(k − 3)$ for $k = 5$ to 18.

In general, given that the first roll yields outcome $n$, we want to compute $dn(k) \cdot d2(k − n)$ for $k = 2 + n$ to $6n$.

$$>
\left( \sum_{k=2+1}^{6(1)} d1_k d2_{k-1} \right) + \left( \sum_{k=2+2}^{6(2)} d2_k d2_{k-2} \right) + \left( \sum_{k=2+3}^{6(3)} d3_k d2_{k-3} \right) + \left( \sum_{k=2+4}^{6(4)} d4_k d2_{k-4} \right) + \left( \sum_{k=2+5}^{6(5)} d5_k d2_{k-5} \right) + \left( \sum_{k=2+6}^{6(6)} d6_k d2_{k-6} \right)
$$

$$>
\frac{320573}{839808}
$$

Since the outcomes of the first roll are equiprobable, the probability of a tie is 1/6 of this value.

$$>	ext{Tie} = \frac{320573}{5038848}
$$

**Fourth part.** What is the probability Player A beats Player B? Again we condition on the outcome of the first roll.

If the 1st roll yields 1, then, as before, Player A may obtain any value from 2+1 to 12+1 and Player B only values from 1 to 6. If Player B obtains, say, the outcome $k$, then Player A wins with any outcome from $k + 1$ to 13. This requires that on the remaining two throws that Player A obtain at least $k$. The probability of this event is clearly $d1(k) \cdot (d2(k) + d2(k + 1)) + \ldots$.

If the 1st roll should yield a 3, Player A can obtain any value from 2+3 to 12+3 and B any value from 3 to 18. If B threw a 12, for example, then A would need at least a 10 from the remaining 2 dice. In general, if B throws $k$, A needs at least $k − 2$ from remaining dice so the probability that A beats B must be $d3(k) \cdot (d2(k − 2) + d2(k − 1)) + \ldots$.

$$>
\left( \sum_{k=1}^{6(1)} \sum_{j=k}^{36} d1_k d2_j \right) + \left( \sum_{k=2}^{6(2)} \sum_{j=k-1}^{36} d2_k d2_j \right) + \left( \sum_{k=3}^{6(3)} \sum_{j=k-2}^{36} d3_k d2_j \right) + \left( \sum_{k=4}^{6(4)} \sum_{j=k-3}^{36} d4_k d2_j \right) + \left( \sum_{k=5}^{6(5)} \sum_{j=k-4}^{36} d5_k d2_j \right) + \left( \sum_{k=6}^{6(6)} \sum_{j=k-5}^{36} d6_k d2_j \right)
$$

$$>
\frac{215555}{93312}
$$

Again, since the outcomes of the first cast are equiprobable, the probability that A beats B is 1/6 of this.
value.

> Beat := %/6;

\[
\text{Beat} = \frac{215555}{559872}
\]

Player A loses to Player B if his total is less than that of A.

For example, if the first toss is 1, then Player B may throw from 1 to 6. Now since Player A is able to throw from 2+1 to 12+1, B cannot win with a 1 or a 2. Moreover, if B throws a 3, he can only tie. Therefore, only \( k = 4 \) to 6 are potentially winning throws. In this case, on the remaining casts, Player A will lose if he throws at most \( k - 2 \). This occurs with probability \( d_2(k - 2) + d_2(k - 3) + \ldots d_2(1) \).

If the first toss should yield a 3, Player B can produce any number between 3 and 18 and Player A from 2+3 to 12+3. Since A has minimum score 5, B cannot win with a 3 or 4 and can only tie with a 5. Thus, only outcomes 6 to 18 are of concern. If Player B casts, say, 11, Player A will lose if the remaining two throws produce at most 7. This occurs with probability \( d_2(7) + d_2(6) + \ldots \). In general, if B rolls \( k \), the probability of A losing is \( d_2(k - 4) + d_2(k - 5) + \ldots d_2(1) \).

In complete generality, if the first cast is \( n \), only the outcomes \( n + 3 \) to 6 \( n \) need be considered. If then B rolls \( k \), Player A can produce at most \( k - n - 1 \).

> \[
\sum_{k=4}^{6} \left( \sum_{j=1}^{k-2} d_{1_k} d_{2_j} \right) + \sum_{k=5}^{6} \left( \sum_{j=1}^{k-3} d_{2_k} d_{2_j} \right) + \sum_{k=6}^{6} \left( \sum_{j=1}^{k-4} d_{3_k} d_{2_j} \right) + \sum_{k=7}^{6} \left( \sum_{j=1}^{k-5} d_{4_k} d_{2_j} \right) + \sum_{k=8}^{6} \left( \sum_{j=1}^{k-6} d_{5_k} d_{2_j} \right) + \sum_{k=9}^{6} \left( \sum_{j=1}^{k-7} d_{6_k} d_{2_j} \right)
\]

\[
\frac{347285}{104976}
\]

Again, since the outcomes of the first cast are equiprobable, we divide by 6.

> Lose := %/6;

\[
\text{Lose} = \frac{347285}{629856}
\]

Check.

> Lose+Beat+Tie;

1

> Expect_A := Beat+Tie/2;

\[
\text{Expect}_A = \frac{4200563}{10077696}
\]

> Expect_B := Lose+Tie/2;
\[
\frac{5877133}{10077696}
\]

\[
\frac{4200563}{5877133}
\]

\[
0.7147299542
\]

Problem VI.

\[
\frac{282571}{629856}
\]

\[
\frac{347285}{629856}
\]

And Bernoulli's solution agrees with this.