RECHERCHES

SUR

L'INTEGRATION DES ÉQUATIONS DIFFÉRENTIELLES

AUX DIFFÉRENCES FINIES

ET SUR

LEUR USAGE DANS LA THÉORIE DES HASARDS

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Mémoires de l'Académie Royale des Sciences de Paris
(Savants étrangers), 1773, T. VII (1776) pp. 37-162.
Oeuvres Complète **8**, pp. 144-197.

Read to the Academy 10 February 1773.

XXV.

Application of the preceding researches to the analysis of chances.

The present state of the system of Nature is evidently a sequel of that which was in the preceding moment, and, if we imagine an intelligence who, for a given instant, embraces all the relationships of the beings of this universe, she could determine for any time taken in the past or in the future the respective position, the movements, and generally the attachments of all these beings.

Physical astronomy, this of all our attainments which gives the greatest credit to the human spirit, offers us an idea, although imperfect, of that which could be a similar intelligence. The simplicity of the law which moves the celestial bodies, the relationships of their masses and of their distances, permits the analysis to follow, up to a certain point, their movements; and, in order to determine the state of the system of these great bodies in the past or future centuries, it suffices to the geometer that observation gives to him their position and their velocity for any instant: man owes then this advantage to the power of the instrument which he employs, and to the small number of relationships which he embraces in his calculations; but the ignorance of the different

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causes which compete in the production of events, and their complication, joining to the imperfection of the analysis, prevents pronouncing with the same certitude on the great number of phenomena; there are for him therefore some uncertain things, these are more or less probable. In the impossibility to know them, he has sought to compensate himself by determining their different degrees of possibility, so that we owe to the feebleness of the human mind one of the most delicate and most ingenious theories of Mathematics, known as the science of chances or of probabilities.

Before going further, it is important to fix the sense of these words *chance* and *probability*. We regard a thing as the effect of chance, when it offers to our eyes no regularity, or which announces no design, and when we are ignorant moreover of the causes which have produced it. Chance has therefore no reality in itself; it is only a proper term to designate our ignorance of the manner in which the different parts of a phenomenon are coordinated among themselves and with the rest of Nature.

The notion of probability depends upon our ignorance. If we are assured that, of two events which cannot exist together, one or the other must necessarily happen, and if we see no reason in order that one would happen rather than the other, the existence and the nonexistence of each of them is equally probable. Similarly, if of three events which are mutually exclusive, one must necessarily happen, and if we see no reason in order that one would happen rather than the other, their existence is equally probable, but the nonexistence of each of them is more probable than its existence, and this in the ratio of 2 to 1, because on three equally probable cases there are two which are favorable to it, and one alone which is contrary to it.

The number of possible cases remaining the same, the probability of an event increases with the number of favorable cases; on the contrary, the number of favorable cases remaining the same, it diminishes in measure as the number of possible cases increases; so that it is in direct proportion to the number of favorable cases and in inverse to the number of all the possible cases.

The probability of the existence of an event is thus only the ratio of the number of favorable cases to that of all the possible cases, when we see moreover no reason in order that one of these cases would happen rather than the other. It can be consequently represented by a fraction of which the numerator is the number of favorable cases, and the denominator that of all the possible cases.

Similarly, the probability of the nonexistence of an event is the ratio of the number of the cases which are contrary to it to that of all the possible cases, and must be consequently expressed by a fraction of which the numerator is the number of contrary cases, and the denominator that of all the possible cases.

It follows thence that the probability of the existence of an event added to the probability of its nonexistence makes a sum equal to unity which represents consequently entire certitude, because it is clear that an event must necessarily either rightly happen or fail.

Moreover, a thing happens certainly when all the possible cases are favorable to it, and the fraction which expresses its probability is then unity itself. Certitude can therefore be represented by the unit, and probability by a fraction of certitude; it can approach more and more to unity, and even differ from it less than any given quantity; but it can never become greater. The theory of chances has for object to determine these fractions, and one sees thence that it is the most happy supplement that one can imagine to the uncertainty of our knowledge.

Certitude and probability, such as we just defined them, are evidently comparable between them and can be subjected to a rigorous calculus; it is not therefore some different states of the human mind when it sees only all the possible cases favoring an event, or when, in this number, it realizes many of them which are contrary to it. These two states are absolutely incomparable, and one cannot say of the first that it is the double, or triple of the second, because truth is indivisible. There happens here the same thing as in all the physico-mathematical sciences; we measure the intensity of light, the different degrees of heat of bodies, their forces, their resistances, etc. In all these researches, the physical causes of our sensations, and not the sensations themselves, are the object of Analysis.

The problem of events serves to determine the expectation or the fear of the persons interested in their existence, and it is under this point of view that the science of chances is one of the most useful of the civil life. This word *expectation* has different meanings: it ordinarily expresses the state of the human mind when there must happen to it any good under certain assumptions which are only probable. In the theory of chances, expectation is the product of the expected sum by the probability to obtain it. In order to distinguish the two meanings of this term, I will call the first *moral expectation*, and the second, *mathematical expectation*.

We imagine n persons who have an equal probability to obtain the sum a, and that this sum must certainly belong to one among them; the total probability being 1, or equal to certitude, it is clear that the probability of each of these persons is $\frac{1}{n}$, and consequently their mathematical expectation $\frac{a}{n}$. This is thus the sum which ought to return to them, if they wished, without incurring the risks of the events, sharing the entire sum a.

If one of these persons p had a probability double of that of the others, his mathematical expectation and, consequently, the sum which ought to return to him in the sharing would be similarly two times greater; because, if one imagines $n+1$ persons who have an equal probability on the sum a, their probability to obtain it will be $\frac{1}{n+1}$, and their mathematical expectation $\frac{a}{n+1}$. Now one can suppose that one among them cedes his claims and his expectation to p; this one will acquire consequently a double probability and a double expectation expressed by $\frac{2a}{n+1}$; and in the sharing he must have a sum $\frac{2a}{n+1}$ double of that of the other persons.

We see thence that the mathematical expectation is nothing other than the partial sum which must be returned when one does not wish to incur the risks of the event, by supposing that the apportionment of the entire sum is made proportionally to the probability to obtain it; it is in fact the only equitable manner to apportion it when we set aside all strange circumstances, because with an equal degree of probability one has an equal right to the expected sum.

Moral expectation depends, in this way as the mathematical expectation, on the expected sum and on the probability to obtain it; but is not always proportional to the product of these two quantities; it is ruled by a thousand variable circumstances, that it is nearly always impossible to define, and even more to subject to Analysis; these circumstances, it is true, serve only to increase or to diminish the advantage that procures the expected sum, and so we can regard the moral expectation itself as the product of this advantage by the probability to obtain it; but we must distinguish, in the expected good, its relative value to its absolute value; this here is absolutely independent of the need and of the other reasons which make it wished for, instead of which the first increases with these different motives.

Now we cannot give any determinate rule to appreciate this relative value; there is however a most ingenious point that Mr. Daniel Bernoulli proposes in the Volume of Petersburg for the year 1730. The relative value of a very small sum is, according to this illustrious geometer, proportional to its absolute value divided by the total wealth of the interested person.

This rule is however not general, but it must serve in a great number of circumstances, and it is all that one can desire in this matter.

Most of that which was written on chances has seemed to confuse expectation and moral probability with expectation and mathematical probability, or to regulate at least one by the other; they have wanted thus to give to their theories an extent to which they are not susceptible, this has rendered them obscure and little fit to satisfy the mind accustomed to the rigorous clarity of Geometry. Mr. d'Alembert has proposed against them some very fine objections, which have awakened the attention of the geometers; he has made felt the absurdity which it would have lead them, in a great number of circumstances, after the results of the Calculus of Probabilities, and, consequently, the necessity to establish in these matters a distinction between the mathematical and the moral; this part of the sciences owes to him therefore the advantage to be supported hereafter on some clear principles and to be tightened in its true bounds.

Let one permit me here the following digression on the difficulties of which the analysis of chances has seemed susceptible: the probability of uncertain things and the expectation which is found linked to their existence are, as I have said, the two objects of this Analysis; the distinction established previously between moral expectation and mathematical expectation responds, it seems to me, to all the objections that one could make against the second of these two objects; we examine consequently those which have relationship to the first.

In the research of the probability of events, one starts from this principle, namely that the probability is the number of favorable cases divided by those of all the possible cases, this is evident; there therefore can be difficulty only as much as one would assume an equal possibility to two unequally possible cases; now we cannot be prevented from agreeing that the applications that have been made hitherto of the Calculus of the Probabilities to the objects of civil life are subject to this difficulty. I suppose, for example, that in the game of *heads* and of *tails* the piece that one casts into the air has greater inclination to fall back on one side than the other, but that the two players are unaware of which side has the greatest inclination; it is clear that there are equal odds for *heads* as for *tails*; one can therefore assume on the first toss, as one does ordinarily, that *heads* and *tails* are equally probable; but this assumption is no longer permitted if, for example, one of the players wagers that on two tosses he will bring about *heads*; because then one must take into consideration the possible inequality of *heads* and of *tails*, since, just as one is unaware on what side is found the greatest, however this inequality encourages always the one who wagers that *heads* will not occur in two tosses, in such a way that its probability is greater than if *heads* and *tails* were equally possible; the cause of the error into which one falls comes from this that one assumes equally possible these four cases: 1° *heads* on the first toss, *heads* on the second, that which I designate in this manner (*heads, heads*); 2° (*heads, tails*); 3° (*tails, heads*); 4° (*tails, tails*), that which is not; because these two here (*heads, heads*), (*tails, tails*), are more probable than the two others; in fact, I suppose that $\frac{1+\varpi}{2}$ represents the probability of a side which has the greatest inclination, and $\frac{1-\varpi}{2}$ that of the other side; this put, the probability of (*heads, heads*) will be $\frac{1+2\varpi+\varpi^2}{4}$ if *heads* were the most probable, and $\frac{1-2\varpi+\varpi^2}{4}$ if it were the least probable; but, as there is no more reason to suppose it the one rather than the other, it is necessary to add together these two probabilities and by taking the mean, which gives $\frac{1+\varpi^2}{4}$ for the probability of (*heads, heads*), and hence likewise for that of (*tails, tails*); one will find similarly the probability of (*heads, tails*), or of (*tails, heads*), equal to $\frac{1-\varpi^2}{4}$; one sees therefore that these four cases are not equally possible, and that the inequality of the probabilities of *heads* and of *tails*, provided that one is unaware of what side has the greatest, favors the player who wagers that on two tosses *heads* will not occur.

This which I just said of the game of *heads* and of *tails* is able to be applied to the game of dice, and generally to all the games in which the different events are susceptible to one physical inequality; but, having developed besides this remark with enough extension (*see* in Volume VI of the *Savants étranges* a Memoir *Sur la probabilité des causes par les événements*), I will observe only that, even if one is unaware which are the most probable of these events, however there occurs this of the remarkable, namely, that one can, in nearly all cases, determine to which of the players this inequality is advantageous.

The Theory of chances supposes again that if *heads* and *tails* are equally possible, it will be likewise for all the combinations of them (*heads, heads, heads,* etc.), (*tails, heads, tails,* etc.), etc. Many philosophers have thought that this assumption is incorrect, and that the combinations in which an event occurs many times in sequence are less possible than the others; but it would be necessary to assume for this that the past events have some influence on those which must occur, which is not admissible. I admit, the ordinary march of nature is to intermingle the events, but this comes, it seems to me, from this that the combinations where they are mixed are much more numerous. Here is, however, a specious difficulty, to which it is good to respond. If *heads* happened, for example, twenty times in sequence, one could be quite tempted to believe that this is not the effect of chance, while if *heads* and *tails* were intermingled in any manner, one would not seek the cause. Now, why this difference between these two cases, if it is only because the one is physically less possible than the other? To this, I respond generally that, there where we perceive the symmetry, we believe always to recognize the effect of a cause acting with order, and we reason by this consistently with probabilities, because, a symmetric effect must be necessarily the effect of chance or the one of a regular cause, the second of these assumptions is more probable than the first. Let $\frac{1}{m}$ be the probability of its existence in the case where it would be due to chance, and $\frac{1}{n}$ this probability if it started from a regular cause; the probability of the existence of this cause will be (*see* Volume VI of *Savants étranges*)

$$\frac{\frac{1}{n}}{\frac{1}{m}+\frac{1}{n}} = \frac{1}{1+\frac{n}{m}};$$

whence one sees that the more m will be great with respect to n, the more also the probability that the symmetric event is the effect of a regular cause will increase. This is not because the symmetric event is less possible than the others, but because there is greater odds that it is due to a cause acting with order than to pure chance, that we seek this cause. A quite simple example will clarify this remark. I suppose that one finds on a table some printed characters arranged in this order, INFINITÉSIMAL; the reason which leads us to believe that this arrangement is not the effect of chance can come only from this that, physically speaking, it is less possible than the others, because, if the word *infinitésimal* were not used in any language, this arrangement would be neither greater, nor less possible, and yet we would suspect then no particular cause. But, as this word is in use among us, it is incomparably more probable that a person will have thus arranged the preceding characters, than it is only that this disposition is due to chance. I return now to my object.

The uncertainty of human knowledge carries either on the events, or on the cause of the events. If we are assured, for example, that an urn contains only some black and white tickets in a given ratio, and that we ask the probability that by taking at random one of these tickets it will be white, the event is uncertain, but the cause on which depends the probability of its existence, that is to say the ratio of the white tickets to the black, is known.

In the following problem: *An urn being supposed to contain a given number of black and white tickets, if one draws from it a white ticket, to determine the probability that the proportion of the white tickets to the black in the urn is that of p to q;* the event is known and the cause unknown.

We can restore to these two classes of problems all those which depend on the Theory of chances. There exists, in truth, a very great number in which the cause and the event seem equally unknown; such is the one: *An urn being supposed able equally to contain all the numbers of white and black tickets from 2 to n inclusively, to determine the probability that by drawing at random two of these tickets, they will be white.* The ratio of the white tickets to the black, the total number of tickets and the event which must result from it are unknown; but one must regard here as cause of the event the equal possibility of all the numbers from 2 to n, and the indifference of the tickets to be white or black; thus this problem is of the genre of those in which, the cause being known, the event is unknown.

My design being not to give here a complete treatise on the Theory of chances, I will be content to apply the preceding researches to the solution of many problems related to this Theory; I will limit myself even here to those in which, the cause being known, the question is to determine the events, having considered in one other Memoir the case where one proposes to reascend again from the events to the causes (*see* Volume VI of *Savants étrangers*).

XXVI.

PROBLEM X. — *If in a pile of x pieces one takes a number at random, it is necessary to determine the probability that this number be even or odd.*

I suppose that we can take indifferently, or one alone, or many, or all these pieces at one time.

This put, let y_x be the sum of the cases in which the number can be even, and ${}^1y_x$ that of the cases in which it can be odd; it is clear that, if we increase the number x of pieces by one unit, the sum of the even cases, represented thus by y_{x+1} will be equal: 1° to the preceding number of even cases; 2° to the preceding number of odd cases, since each of these cases, combined with the new piece, give an even case. We will have therefore

$$y_{x+1} = y_x + {}^1y_x; \tag{1}$$

next the number of odd cases, represented by ${}^1y_{x+1}$ will be equal: 1° to the preceding number ${}^1y_x$ of odd cases; 2° to the preceding number of even cases; 3° to the unit, since the new piece can be taken alone. We will have consequently

$${}^1y_{x+1} = {}^1y_x + y_x + 1. \tag{2}$$

In order to integrate these two equations, I observe that the equation (1) gives

$$\Delta y_x = {}^1y_x \quad \text{hence,} \quad \Delta^2 y_x = \Delta.{}^1y_x.$$

Now equation (2) gives

$$\Delta.{}^1y_x = y_x + 1, \quad \text{therefore} \quad \Delta^2 y_x = y_x + 1;$$

whence it is easy to conclude

$$y_{x+1} = 2y_x + 1,$$

By integrating this equation by Problem I, we will have

$$y_x = A2^x - 1,$$

A being an arbitrary constant; in order to determine it, I observe that, x being 1, we have

$$y_x = 0, \quad \text{therefore} \quad A = \frac{1}{2}, \quad \text{hence} \quad y_x = 2^{x-1} - 1.$$

Now, since we have ${}^1y_x = \Delta y_x$, we will have ${}^1y_x = 2^{x-1}$. The sum of all the possible cases is clearly

$$y_x + {}^1y_x = 2^x - 1.$$

If therefore we call z_x the probability that the number of pieces is even, and ${}_1z_x$ that it is odd, we will have

$$z_x = \frac{2^{x-1} - 1}{2^x - 1} \quad \text{and} \quad {}_1z_x = \frac{2^{x-1}}{2^x - 1};$$

whence there results that there is always more advantage to wager for the odd numbers than for the evens.

I suppose that one is assured that the number x cannot exceed the number n, but that this number and all the lesser are equally possible, we will have the sum of all the odd cases $= 2^x + C$. Now, x being 1, we must have $2^x + C = 1$; therefore $C = -1$. We will find similarly the sum of all the even cases $= 2^x - x + C$; now, x being 1, we have $2^x - x + C = 0$. Therefore $C = -1$; hence, the sum of the odd cases is $2^n - 1$, and the sum of the even cases is $2^n - n - 1$; thus, the probability for the odds is

$$\frac{2^n - 1}{2^{n+1} - n - 2},$$

and the probability for the evens

$$\frac{2^n - n - 1}{2^{n+1} - n - 2}.$$

XXVII.

PROBLEM XI. — *Let a be a sum which Paul constitutes to an annuity, in a way that the interest is $\frac{1}{m}$ of that which is due to him: I suppose that, for some arbitrary reasons, one keeps each year the fraction $\frac{1}{n}$ of this interest, so that Paul, at the end of the first year, for example, must collect only the quantity $\frac{a}{m} - \frac{a}{mn}$, this put, if one pays to him every year the sum $\frac{a}{m}$, and, consequently, more than is due to him, and let the surplus be used to amortize the capital, one asks what this capital will become in the year x.*

Let y_x be this capital in the year x; it is clear that, at the end of the year x, there will be due to Paul only $y_x \left(\frac{1}{m} - \frac{1}{mn}\right)$. Therefore, since one pays the sum $\frac{a}{m}$, the capital will be diminished by the quantity $\frac{a}{m} - y_x \frac{n-1}{mn}$; hence, we will have

$$y_{x+1} = y_x - \frac{a}{m} + y_x \frac{n-1}{mn}$$

and, integrating as in Problem I,

$$y_x = \frac{na}{n-1} + A\left(1 + \frac{n-1}{mn}\right)^{x-1};$$

now, setting $x = 1$, $y_x = a$; thus,

$$A = -\frac{a}{n-1};$$

hence,

$$y_x = \frac{a}{n-1}\left[n - \left(1 + \frac{n-1}{mn}\right)^{x-1}\right].$$

If we ask the year x at which this capital will be zero, we will have

$$\left(1 + \frac{n-1}{mn}\right)^{x-1} = n;$$

therefore

$$x = 1 + \frac{\ln n}{\ln\left(1 + \frac{n-1}{mn}\right)}.$$

I suppose that the interest be 5 for 100, and that one collects $\frac{1}{10}$ on this interest, we will have

$$m = 20 \quad \text{and} \quad n = 10;$$

hence,

$$x = 53.3.$$

One can resolve in the same manner the following problem:

A person owes the sum a, and wishes to release himself at the end of h years, so that she owes nothing in the year $h+1$, the interest being always $\frac{1}{m}$ of the quantity due; the question is to find what must she give for this each year.

Let p be this quantity, and y_x that which she owes in year x, we will have, by the preceding method,

$$y_{x+1} = y_x \left(1 + \frac{1}{m}\right) - p,$$

whence I conclude by integrating $y_x = mp + A\left(1 + \frac{1}{m}\right)^{x-1}$. Now, putting $x = 1$, $y_x = a$; thus

$$a = mp + A;$$

hence,

$$y_x = mp + (a - mp)\left(1 + \frac{1}{m}\right)^{x-1};$$

but, by making $x = h+1$, we have

$$y_x = 0,$$

by assumption; therefore

$$p = \frac{a\left(1 + \frac{1}{m}\right)^h}{m\left[\left(1 + \frac{1}{m}\right)^h - 1\right]}.$$

XXVIII.

PROBLEM XII. — *I imagine a solid composed of a number n of perfectly equal faces, and which I designate by the numbers* $1, 2, 3, \ldots, n$*; I wish to have the probability that, in a number x of casts, I will bring about these n faces in sequence in the order* $1, 2, 3, 4, \ldots, n$.

I call y_x this probability, and u_x the number of favorable cases: the number of all the possible cases is n^x; because, if we call t_x this number at the cast x, it will be t_{x-1} at the cast $x-1$. Now, the number of cases at the cast $x-1$ must be combined with all the faces of the solid, in order to form all the possible cases at the cast x; we have therefore

$$t_x = nt_{x-1},$$

this which gives

$$t_x = An^x.$$

Now, setting $x=1$, $t_x=n$; thus

$$A=1 \quad \text{and} \quad t_x=n^x.$$

We will have therefore

$$\frac{u_x}{n^x}=y_x.$$

Now u_x is evidently equal to the number of favorable cases at the cast $x-1$ multiplied by the number of faces of the solid, plus to the number of cases in which the combination $1, 2, 3, \ldots, n$ can happen precisely at the cast x; moreover, all the cases in which this combination does not happen at the cast $x-n$ each gives a case in which it will happen precisely at the cast x. The number of these cases is $n^{x-n}-u_{x-n}$; we will have therefore

$$u_x=nu_{x-1}+n^{x-n}-u_{x-n}; \quad \text{hence,} \quad y_x=y_{x-1}-\frac{y_{x-n}}{n^n}+\frac{1}{n^n},$$

an equation which we will integrate easily by the preceding methods.

Let $n=2$: we will have

$$y_x=y_{x-1}-\frac{y_{x-2}}{4}+\frac{1}{4};$$

whence I conclude, by integrating,

$$y_x=1+\frac{Ax+B}{2^{x-1}};$$

now, setting $x=1$, $y_x=0$, and setting $x=2$, $y_x=\frac{1}{4}$; thus, $A=-\frac{1}{2}$, and $B=-\frac{1}{2}$; hence, $y_x=1-\frac{x+1}{2^x}$.

XXIX.

PROBLEM XIII. — *I suppose a number n of players $(1), (2), (3), \ldots, (n)$ play in this way: (1) plays with (2), and if he wins he wins the game; if he neither loses nor wins, he continues to play with (2), until one of the two wins. But if (1) loses, (2) plays with (3); if he wins it, he wins the game; if he neither loses nor wins, he continues to play with (3); but if he loses, (3) plays with (4), and thus in sequence until one of the players has defeated the one who follows him; that is to say (1) must be winner over (2), or (2) over (3), or (3) over (4), $\ldots$, or $(n-1)$ over (n), or (n) over (1). Moreover, the probability of anyone of the players to win over the other equals $\frac{1}{3}$, and that of neither winning nor losing equals $\frac{1}{3}$. This put, it is necessary to determine the probability that one of these players will win the game at trial x.*

Let $\overset{n}{u}_x$ be the probability that at trial x, (n) will be the winner over $(n-1)$: we will have

$$\overset{n}{u}_x=\frac{1}{3}\overset{n}{u}_{x-1}+\frac{1}{3}\overset{n-1}{u}_{x-1}$$

Let now $\overset{1}{z}_x$ be the probability that (n), at trial x, will win the game, $\overset{2}{z}_x$ the probability that it will be $(n-1)$, and thus in sequence: we will have $\overset{1}{z}_x=\frac{1}{3}\overset{n}{u}_{x-1}$. Hence,

$$\overset{1}{z}_x-\frac{1}{3}\overset{1}{z}_{x-1}=\frac{1}{3}\overset{2}{z}_{x-1}.$$

We will have likewise

$$\overset{2}{z}_x - \frac{1}{3}\overset{2}{z}_{x-1} = \frac{1}{3}\overset{3}{z}_{x-1},$$
$$\overset{3}{z}_x - \frac{1}{3}\overset{3}{z}_{x-1} = \frac{1}{3}\overset{4}{z}_{x-1},$$
$$\vdots$$

such that these equations are reentrant. This put, by following the method set forth previously for this type of equations, we will have

$$\overset{1}{z}_x - \frac{2}{3}\overset{1}{z}_{x-1} + \frac{1}{3^2}\overset{1}{z}_{x-2} = \frac{1}{3}(\overset{2}{z}_{x-1} - \frac{1}{3}\overset{2}{z}_{x-1}) = \frac{1}{3^2}\overset{4}{z}_{x-3};$$

hence,

$$\overset{1}{z}_x - \frac{3}{3}\overset{1}{z}_{x-1} + \frac{3}{3^2}\overset{1}{z}_{x-2} - \frac{1}{3^3}\overset{1}{z}_{x-3} = \frac{1}{3^3}(\overset{3}{z}_{x-2} - \frac{1}{3}\overset{3}{z}_{x-2}) = \frac{1}{3^3}\overset{4}{z}_{x-3};$$

whence, by continuing to operate so, we will have

$$\overset{1}{z}_x - \frac{n}{3}\overset{1}{z}_{x-1} + \frac{n(n-1)}{1.2}\frac{1}{3^2}\overset{1}{z}_{x-2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{3^3}\overset{1}{z}_{x-3} + \cdots = \frac{1}{3^n}\overset{1}{z}_{x-n};$$

we will have similarly

$$\overset{2}{z}_x - \frac{n}{3}\overset{2}{z}_{x-1} + \frac{n(n-1)}{1.2}\frac{1}{3^2}\overset{2}{z}_{x-2} - \frac{n(n-1)(n-2)}{1.2.3}\frac{1}{3^3}\overset{2}{z}_{x-3} + \cdots = \frac{1}{3^n}\overset{2}{z}_{x-n},$$

and thus in sequence for the other variables $\overset{3}{z}_x, \overset{4}{z}_x, \ldots$

In order to integrate these different equations, it is necessary to solve this here $(f - \frac{1}{3})^n = \frac{1}{3^n}$; or, by making $f - \frac{1}{3} = q$, $q^n - \frac{1}{3^n} = 0$, this which is easy to do, by the beautiful theorem of Cotes. There remains in this way no more difficulty than the determination of the arbitrary constants which come from the integration. For this, it is necessary to have the probability of winning of each player for a number n of trials. Now, for that which regards player (1), his probability of winning on the first trial is $\frac{1}{3}$; on the second trial it is $\frac{1}{3^2}$; on the third trial it is $\frac{1}{3^3}$, ..., so that we have

$$\begin{array}{cccccc} 1, & 2, & 3, & 4, & \ldots, & n, \\ \frac{1}{3}, & \frac{1}{3^2}, & \frac{1}{3^3}, & \frac{1}{3^4}, & \ldots, & \frac{1}{3^n}, \end{array}$$

by setting under each trial the probability of player (1) winning at this trial; we will form likewise for player (2) the sequence

$$\begin{array}{cccccc} 2, & 3, & 4, & 5, & \ldots, & n+1, \\ \frac{1}{3^2}, & \frac{2}{3^3}, & \frac{3}{3^4}, & \frac{4}{3^5}, & \ldots, & \frac{n}{3^{n+1}}, \end{array}$$

and for player (3) this one:

$$\begin{array}{cccccc} 3, & 4, & 5, & 6, & \ldots, & n+2, \\ \frac{1}{3^3}, & \frac{3}{3^4}, & \frac{6}{3^5}, & \frac{10}{3^6}, & \ldots, & \frac{\frac{n(n+1)}{1.2}}{3^{n+2}}, \end{array}$$

and thus in sequence for the other players.

XXX.

PROBLEM XIV. — *Two players* A *and* B*, of whom the respective skills are in ratio of p to q, play together in a way that, out of a number x of trials, there lacks n of them to player* A*, and consequently $x-n$ to player* B*, in order to win; the question is to determine the respective probabilities of these two players.*

Let ${}_n y_x$ be the probability of B winning; it is clear that on the following trial it will be, either ${}_{n-1}y_{x-1}$, if B loses, or ${}_n y_{x-1}$, if he wins. Now, the probability that he will win is $\frac{q}{p+q}$, and that he will lose, $\frac{p}{p+q}$. We have therefore

$$(g) \qquad {}_n y_x = \frac{q}{p+q}\,{}_n y_{x-1} + \frac{p}{p+q}\,{}_{n-1}y_{x-1}.$$

This equation is in partial differences. In order to integrate I observe that, when $n=1$, we have ${}_1y_x = \frac{q}{p+q}\,{}_1y_{x-1}$, since in this case ${}_{n-1}y_x = 0$; we will have therefore by Problem VI, article XX,

$${}_n y_x = a_n \cdot {}_n y_{x-1} + {}^1a_n \cdot {}_n y_{x-2} + {}^1a_n \cdot {}_n y_{x-3} + \cdots + u_n,$$

and we will find that the equation

$$0 = 1 - \frac{a_n}{f} - \frac{{}^2a_n}{f} - \cdots$$

is the same as this one:

$$0 = \left(f - \frac{q}{p+q}\right)^n.$$

We will have besides $u_n = \frac{p}{p+q}u_{n-1}$, therefore $u_n = H\left(\frac{p}{p+q}\right)^n$. Now, setting $n=1$, $u_n = 0$; thus $H=0$, and $u_n = 0$. The expression of ${}_n y_x$ will be therefore (art. IX)

$$\begin{aligned} {}_n y_x = \frac{q^{x-1}}{(p+q)^{x-1}} \Bigg[& C_n + D_n(x-1) + E_n\frac{(x-1)(x-2)}{1.2} + \cdots \\ & + L_n\frac{(x-1)(x-2)\cdots(x-n+1)}{1.2.3\cdots(n-1)}\Bigg]. \end{aligned}$$

In order to determine the arbitrary constants $C_n, D_n, E_n, \ldots$, which can be functions of n, I observe that, if one makes $x=n$, we will have ${}_n y_n = 1$; because it is clear that A loses necessarily, when out of n trials there lacks n of them to him; if one makes $x = n-1$, we will have similarly ${}_n y_{n-1} = 1$; because equation (g) gives

$${}_n y_n = \frac{q}{p+q}\,{}_n y_{n-1} + \frac{p}{p+q}\,{}_{n-1}y_{n-1}$$

or

$$1 = \frac{q}{p+q}\,{}_n y_{n-1} + \frac{p}{p+q},$$

hence ${}_{n}y_{n-1} = 1$; similarly, if one makes $x = n-2$, we will have ${}_{n}y_{n-2} = 1$, and so in sequence. If therefore one makes in the expression of ${}_{n}y_{x}$, $x = 1$, we will have ${}_{n}y_{1} = 1$; hence, $C_n = 1$. If one makes $x = 2$, we will have

$$1 = (C_n + D_n)\frac{q}{p+q};$$

hence, $D_n = \frac{p}{q}$. If one makes $x = 3$, we will have

$$1 = (C_n + 2D_n + E_n)\frac{q^2}{(p+q)^2} = (1 + 2\frac{p}{q} + E_n)\frac{q^2}{(p+q)^2},$$

therefore $E_n = \frac{p^2}{q^2}$, and thus in sequence; whence it is easy to conclude

$${}_{n}y_{x} = \frac{1}{(\frac{p}{q}+1)^{x-1}}\left[1 + \frac{p}{q}(x-1) + \frac{p^2}{q^2}\frac{(x-1)(x-2)}{1.2} + \frac{p^3}{q^3}\frac{(x-1)(x-2)(x-3)}{1.2.3} + \cdots \right.$$
$$\left. + \frac{p^{n-1}}{q^{n-1}}\frac{(x-1)(x-2)\cdots(x-n+1)}{1.2.3\cdots(n-1)}\right].$$

XXXI.

PROBLEM XV. — *Three players* A, B, C*, of whom the respective abilities are represented by the letters* p, q, r*, play together in a manner that, out of a number* x *of trials, there lacks* m *to* A*,* n *to* B *and* $x-m-n$ *to* C*; one proposes to determine the respective probability of these three players for winning.*

Let ${}_{m,n}y_{x}$ be the probability of C winning; it is clear that after a new trial it will be, either ${}_{m-1,n}y_{x-1}$, or ${}_{m,n-1}y_{x-1}$, or ${}_{m,n}y_{x-1}$; now, the probability that it will be ${}_{m-1,n}y_{x-1}$ is $\frac{p}{p+q+r}$; the probability that it will be ${}_{m,n-1}y_{x-1}$ is $\frac{q}{p+q+r}$; and the probability that it will be ${}_{m,n}y_{x-1}$ is $\frac{r}{p+q+r}$. We will have therefore

$$(o) \qquad {}_{m,n}y_{x} = \frac{p}{p+q+r}\,{}_{m-1,n}y_{x-1} + \frac{q}{p+q+r}\,{}_{m,n-1}y_{x-1} + \frac{r}{p+q+r}\,{}_{m,n}y_{x-1}.t$$

This equation is in partial differences in four variables, and is integrated by Problem IX; but, for this, it is necessary that one have two particular equations for the case of $m = 1$ and of $n = 1$; in order to find them, I observe that, if one makes $m = 1$, we will have

$$(p) \qquad {}_{1,n}y_{x} = \frac{r}{p+q+r}\,{}_{1,n}y_{x-1} + \frac{q}{p+q+r}\,{}_{1,n-1}y_{x-1},$$

because, when $m = 1$, we have ${}_{m-1,n}y_{x-1} = 0$.

Equation (p) is in partial differences in two variables; in order to integrate it, I observe that, if one supposes $n = 1$, we have

$${}_{1,1}y_{x} = \frac{r}{p+q+r}\,{}_{1,1}y_{x-1};$$

from this equation and from equation (p), we will conclude easily, by Problem VI,

$$(q)\qquad \begin{cases} {}_{1,n}y_x = n\dfrac{r}{p+q+r}\,{}_{1,n}y_{x-1} - \dfrac{n(n-1)}{1.2}\dfrac{r^2}{(p+q+r)^2}\,{}_{1,n}y_{x-2} \\ +\dfrac{n(n-1)(n-2)}{1.2.3}\dfrac{r^3}{(p+q+r)^3}\,{}_{1,n}y_{x-3} - \cdots \end{cases}$$

We will have similarly

$$(q')\qquad \begin{cases} {}_{m,1}y_x = m\dfrac{r}{p+q+r}\,{}_{m,1}y_{x-1} - \dfrac{m(m-1)}{1.2}\dfrac{r^2}{(p+q+r)^2}\,{}_{m,1}y_{x-2} \\ +\dfrac{m(m-1)(m-2)}{1.2.3}\dfrac{r^3}{(p+q+r)^3}\,{}_{m,1}y_{x-3} - \cdots \end{cases}$$

By means of these equations and of equation (o), we will determine, by Problem IX, the general expression of ${}_{m,n}y_x$; thus the problem proposed has no other difficulty than the length of the calculation.

The general method of Problem IX leads to one final very elevated equation; but, by means of particular considerations, I have arrived at the solution of the preceding problem by a much simpler method, that I have developed. I have for brevity $p+q+r=1$, and equation (o) gives

$$(o')\qquad {}_{2,n}y_x = p.\,{}_{1,n}y_{x-1} + q.\,{}_{2,n-1}y_{x-1} + r.\,{}_{2,n}y_{x-1},$$

and if one makes $m=2$, equation (q') gives

$${}_{2,1}y_x = 2r\cdot{}_{2,1}y_{x-1} - r^2\cdot{}_{2,1}y_{x-2}.$$

Let

$$(s)\qquad {}_{2,n}y_x = a_n\cdot{}_{2,n}y_{x-1} + {}^1a_n\cdot{}_{2,n}y_{x-2} + \cdots + {}_nX_x;$$

therefore

$$q.\,{}_{2,n-1}y_{x-1} = a_{n-1}q.\,{}_{2,n-1}y_{x-2} + {}^1a_{n-1}q.\,{}_{2,n-1}y_{x-3} + \cdots + q.\,{}_{n-1}X_{x-1}.$$

Substituting into this equation, in place of ${}_{2,n-1}y_{x-2}$, ${}_{2,n-1}y_{x-3}$, ..., their values deduced from equation (o'), we will have

$$\begin{aligned} {}_{2,n}y_x =& (r+a_{n-1}).\,{}_{2,n}y_{x-1} + ({}^1a_n - a_{n-1}r)\,{}_{2,n}y_{x-2} \\ &+ p.\,{}_{1,n}y_{x-1} - a_{n-1}p.\,{}_{1,n}y_{x-2} - \cdots + q.\,{}_{n-1}X_{x-1}, \end{aligned}$$

whence, by comparing with equation (s), we will have:

1° $a_n = a_{n-1} + r$, hence, $a_n = (n+1)r + C$; now, setting $n=1$, $a_n = 2r$; thus, $C=0$.

2° ${}^1a_n = {}^1a_{n-1} - a_{n-1}r$, hence, ${}^1a_n = -\frac{n(n+1)}{1.2}r^2 + C$; now, putting $n=1$, ${}^1a_n = -r^2$; thus, $C=0$.

3° ${}^2a_n = {}^2a_{n-1} + \frac{n(n-1)}{1.2}r^3$; therefore, ${}^2a_n = \frac{(n-1)n(n+1)}{1.2.3}r^3 + C$; now, setting $n = 1$, ${}^2a_n = 0$; therefore, $C = 0$, and thus the rest. Hence,

$$\begin{aligned} &p({}_{1,n}y_{x-1} - a_{n-1}\cdot {}_{1,n}y_{x-2} - \cdots) \\ &\quad = p\left[{}_{1,n}y_{x-1} - nr.{}_{1,n}y_{x-2} + \frac{n(n-1)}{1.2}r^2.{}_{1,n}y_{x-3} - \cdots\right] = 0, \end{aligned}$$

by virtue of equation (q).

4° ${}_nX_x = q.{}_{n-1}X_{x-1}$. Now, we have ${}_1X_x = 0$; therefore, ${}_2X_x = 0$, and generally ${}_nX_x = 0$. We have therefore

$${}_{2,n}y_x = (n+1)r.{}_{2,n}y_{x-1} - \frac{n(n+1)}{1.2}r^2.{}_{2,n}y_{x-2} + \frac{(n-1)n(n+1)}{1.2.3}{}_{2,n}y_{x-3} - \cdots$$

We will have, by an entirely similar process,

$${}_{3,n}y_x = (n+2)r.{}_{3,n}y_{x-1} - \frac{(n+2)(n+1)}{1.2}r^2.{}_{3,n}y_{x-2} + \cdots$$

and generally

$${}_{m,n}y_x = (m+n-1)r.{}_{m,n}y_{x-1} - \frac{(m+n-1)(m+n-2)}{1.2}r^2.{}_{m,n}y_{x-2} + \cdots,$$

an equation of which the integral is

$$\begin{aligned} {}_{m,n}y_x = r^{x-2}\Bigg[&{}_mN_n\frac{(x-2)(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-2)} + {}_mM_n\frac{(x-2)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-3)} \\ &+ {}_mL_n\frac{(x-2)\cdots(x-m-n+3)}{1.2.3\ldots(m+n-4)} + {}_mK_n\frac{(x-2)\cdots(x-m-n+4)}{1.2.3\ldots(m+n-5)} \\ &+ {}_mI_n\frac{(x-2)\cdots(x-m-n+5)}{1.2.3\ldots(m+n-6)} + \cdots + {}_mC_n\Bigg]. \end{aligned}$$

The difficulty consists presently in determining the arbitrary constants ${}_mN_n, {}_mM_n, \ldots,$ which are able to be functions of m and of n.

For this, I assume first $m = 1$, and we will have

$$(\sigma) \quad {}_{1,n}y_x = r^{x-2}\left[{}_1C_n + {}_1D_n(x-2) + {}_1E_n\frac{(x-2)(x-3)}{1.2} + \cdots + {}_1N_n\frac{(x-2)\cdots(x-n)}{1.2.3\ldots(n-1)}\right]$$

Now we have ${}_{1,n}y_{n+1} = 1$, as it is clear, since then no trials lack to player C; I take next the equation

$${}_{1,n}y_x = r.{}_{1,n}y_{x-1} + q.{}_{1,n-1}y_{x-1}.$$

If one makes $x = n+1$, we have

$${}_{1,n}y_{n+1} = 1 = r.{}_{1,n}y_n + q,$$

thus

$${}_{1,n}y_n = \frac{1-q}{r};$$

next

$$ {}_{1,n}y_{n} = \frac{1-q}{r} = r.\,{}_{1,n}y_{n-1} + q\frac{1-q}{r}, $$

thus

$$ {}_{1,n}y_{n-1} = \left(\frac{1-q}{r}\right)^2. $$

We will find similarly

$$ {}_{1,n}y_{n-2} = \left(\frac{1-q}{r}\right)^3, $$

and thus in sequence. This put, if one makes $x = 2$, equation (σ) will give $\left(\frac{1-q}{r}\right)^{n-1} = {}_{1}C_{n}$; if one makes $x = 3$, we will have

$$ \left(\frac{1-q}{r}\right)^{n-2} = r\left[\left(\frac{1-q}{r}\right)^{n-1} + {}_{1}D_{n}\right], $$

therefore

$$ {}_{1}D_{n} = \left(\frac{1-q}{r}\right)^{n-2}\frac{q}{r}. $$

By making $x = 4$, we will have

$$ {}_{1}E_{n} = \left(\frac{1-q}{r}\right)^{n-3}\frac{q^2}{r^2}, $$

and thus in sequence; hence

$$ {}_{1,n}y_{x} = r^{x-2}\left[\frac{q^{n-1}}{r^{n-1}}\frac{(x-2)\cdots(x-n)}{1.2.3\ldots(n-1)} + \frac{q^{n-2}}{r^{n-2}}\frac{1-q}{r}\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)} \right. $$
$$ \left. + \frac{q^{n-3}}{r^{n-3}}\left(\frac{1-q}{r}\right)^2\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)} + \cdots + \left(\frac{1-q}{r}\right)^{n-1}\right] $$

We will have, likewise,

$$ {}_{m,1}y_{x} = r^{x-2}\left[\frac{p^{m-1}}{r^{m-1}}\frac{(x-2)\cdots(x-m)}{1.2.3\ldots(m-1)} + \frac{p^{m-2}}{r^{m-2}}\frac{1-p}{r}\frac{(x-2)\cdots(x-m+1)}{1.2.3\ldots(m-2)} + \cdots\right]. $$

If one substitutes now into equation (o), in place of ${}_{m,n}y_{x}$, its value found above, we will have the following equation

$$
\begin{aligned}
&{}_mN_n \frac{(x-3)(x-4)\cdots(x-m-n)}{1.2.3\ldots(m+n-2)} + ({}_mM_n + {}_mN_n)\frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + ({}_mL_n + {}_mM_n)\frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad = + \frac{p}{r}\,{}_{m-1}N_n \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + \frac{p}{r}\,{}_{m-1}M_n \frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad + \frac{q}{r}\,{}_mN_{n-1} \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} \\
&\quad + \frac{q}{r}\,{}_mM_{n-1} \frac{(x-3)\cdots(x-m-n+2)}{1.2.3\ldots(m+n-4)} + \cdots \\
&\quad + {}_mN_n \frac{(x-3)\cdots(x-m-n)}{1.2.3\ldots(m+n-2)} \\
&\quad + {}_mM_n \frac{(x-3)\cdots(x-m-n+1)}{1.2.3\ldots(m+n-3)} + \cdots,
\end{aligned}
$$

whence we will form the following equations:

$$
\begin{aligned}
{}_mN_n &= \frac{p}{r}\,{}_{m-1}N_n + \frac{q}{r}\,{}_mN_{n-1}, \\
{}_mM_n &= \frac{p}{r}\,{}_{m-1}M_n + \frac{q}{r}\,{}_mM_{n-1}, \\
{}_mL_n &= \frac{p}{r}\,{}_{m-1}L_n + \frac{q}{r}\,{}_mL_{n-1}, \\
&\vdots
\end{aligned}
$$

Now we have

$$ {}_1N_n = \frac{q^{n-1}}{r^{n-1}}; $$

therefore

$$ {}_2N_n = \frac{p}{r}\frac{q^{n-1}}{r^{n-1}} + \frac{q}{r}\,{}_2N_{n-1}, $$

hence

$$ {}_2N_n = \frac{q^{n-1}}{r^{n-1}}\frac{p}{r}(n+C); $$

now, putting $n = 1$, ${}_2N_1 = \frac{p}{r}$; therefore

$$ C = 0. $$

Next

$$ {}_3N_n = \frac{p^2}{r^2}\frac{q^{n-1}}{r^{n-1}}n + \frac{q}{r}\,{}_3N_{n-1}; $$

therefore

$$ {}_3N_n = \frac{q^{n-1}}{r^{n-1}}\left[\frac{p^2}{r^2}\frac{n(n+1)}{1.2} + C\right]; $$

now, putting $n = 1$, ${}_3N_1 = \frac{p^2}{r^2}$; therefore

$$ C = 0, $$

and generally

$$ {}_mN_n = \frac{p^{m-1}q^{n-1}}{r^{m+n-2}}\frac{n(n+1)\cdots(n+m-2)}{1.2.3\cdots(m-1)}. $$

We have next

$$ {}_1M_n = \frac{1-q}{r}\frac{q^{n-2}}{r^{n-2}}; $$

therefore

$$ {}_2M_n = \frac{q}{r}\,{}_2M_{n-1} + \frac{p}{r}\frac{1-q}{r}\frac{q^{n-2}}{r^{n-2}}; $$

hence,

$$ {}_2M_n = \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\frac{1-q}{r}(n-1) + C\frac{q^{n-1}}{r^{n-1}}; $$

now, putting $n = 1$, ${}_2M_n = \frac{1-p}{r}$; therefore

$$ C = \frac{1-p}{r} $$

and

$$ {}_2M_n = \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\frac{1-q}{r}(n-1) + \frac{q^{n-1}}{r^{n-1}}\frac{1-p}{r}. $$

We will have similarly

$$ {}_3M_n = \frac{q^{n-2}}{r^{n-2}}\frac{p^2}{r^2}\frac{1-q}{r}\frac{(n-1)n}{1.2} + \frac{q^{n-1}}{r^{n-1}}\frac{p}{r}\left(\frac{1-p}{r}n + C\right); $$

now, putting $n = 1$, ${}_3M_n = \frac{p}{r}\left(\frac{1-p}{r}\right)$; therefore

$$ C = 0. $$

By continuing to operate so, we will find generally

$$ {}_mM_n = \frac{p^{m-1}q^{n-2}}{r^{m+n-3}}\frac{1-q}{r}\frac{(n-1)n\cdots(n+m-3)}{1.2.3\cdots(m-1)} + \frac{q^{n-1}p^{m-2}}{r^{m+n-3}}\frac{1-p}{r}\frac{n(n+1)\cdots(n+m-3)}{1.2.3\cdots(m-2)}. $$

I will observe here, relative to these expressions for ${}_mN_n$ and for ${}_mM_n$, that

$$ \frac{n(n+1)\cdots(n+m-2)}{1.2.3\cdots(m-1)} = \frac{m(m+1)\cdots(m+n-2)}{1.2.3\cdots(n-1)} $$

and that

$$\frac{n(n+1)\cdots(n+m-3)}{1.2.3\cdots(m-2)} = \frac{(m-1)m\cdots(m+n-3)}{1.2.3\cdots(n-1)};$$

whence there results that the quantities ${}_mN_n$ and ${}_mM_n$ remain the same when one changes p to q, m to n, and reciprocally; this which must be moreover by the nature of the problem. We must say as much of the other quantities ${}_mL_n$, ${}_mK_n$, ...

Presently

$${}_mL_n = \frac{p}{r}\,{}_{m-1}L_n + \frac{q}{r}\,{}_mL_{n-1};$$

now, ${}_1L_n = \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\left(\frac{1-q}{r}\right)^2$; therefore we will have, by integrating,

$${}_2L_n = \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\left(\frac{1-q}{r}\right)^2(n-2) + C\frac{q^{n-2}}{r^{n-2}};$$

now, putting $n = 2$, $m = 2$ and $x = 4$, in the expression found above for ${}_{m,n}y_x$, we have

$${}_{2,2}y_4 = r^2({}_2L_2 + 2.{}_2M_2 + {}_2N_2);$$

therefore, since ${}_{2,2}y_4 = 1$,

$${}_2L_2 = \frac{1}{r^2} - \frac{2p}{r^2}(1-q) - \frac{2q}{r^2}(1-p) - \frac{2pq}{r^2};$$

moreover, C equals ${}_2L_2$ in the expression for ${}_2L_n$.

We will find similarly

$$\begin{aligned}{}_3L_n = {} & \frac{q^{n-3}}{r^{n-3}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^2\frac{(n-2)(n-1)}{1.2}\\ & + \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\,{}_2L_2(n-1)\\ & + C\frac{r^{n-1}}{q^{n-1}},\end{aligned}$$

C being an arbitrary constant; now, putting $n = 1$, ${}_3L_n = \left(\frac{1-p}{r}\right)^2$; therefore

$$C = \left(\frac{1-p}{r}\right)^2;$$

hence,

$$\begin{aligned}{}_3L_n = {} & \frac{q^{n-3}}{r^{n-3}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^2\frac{(n-2)(n-1)}{1.2}\\ & + \frac{q^{n-2}}{r^{n-2}}\frac{p}{r}\,{}_2L_2(n-1)\\ & + \left(\frac{1-p}{r}\right)^2\frac{q^{n-1}}{p^{n-1}},\end{aligned}$$

and generally we will have

$$
\begin{aligned}
{}_mL_n = {} & \frac{q^{n-3}p^{m-1}}{r^{m+n-4}}\left(\frac{1-q}{r}\right)^2 \frac{(n-2)(n-1)\cdots(n+m-4)}{1.2.3\cdots(m-1)} \\
& + \frac{q^{n-2}p^{m-2}}{r^{m+n-4}}\,{}_2L_2\,\frac{(n-1)\cdots(m+n-4)}{1.2.3\cdots(m-2)} \\
& + \frac{q^{n-1}p^{m-3}}{r^{m+n-4}}\left(\frac{1-p}{r}\right)^2 \frac{n\cdots(n+m-4)}{1.2.3\cdots(m-3)}.
\end{aligned}
$$

We have next

$$
{}_2K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p}{r}\left(\frac{1-q}{r}\right)^3 + \frac{q}{r}\,{}_2K_{n-1};
$$

hence,

$$
{}_2K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p}{r}\left(\frac{1-q}{r}\right)^3 (n-3) + C\frac{q^{n-3}}{r^{n-3}};
$$

now, putting $n = 3$, we have

$$
C = {}_2K_3.
$$

Likewise,

$$
{}_3K_n = \frac{q^{n-4}}{r^{n-4}}\frac{p^2}{r^2}\left(\frac{1-q}{r}\right)^3 \frac{(n-3)(n-2)}{1.2} + \frac{q^{n-3}}{r^{n-3}}\frac{p}{r}\,{}_2K_3(n-2) + \frac{q^{n-2}}{r^{n-2}}\,{}_3K_2,
$$

and generally we will have

$$
\begin{aligned}
{}_mK_n = {} & \frac{q^{n-4}p^{m-1}}{r^{m+n-5}}\left(\frac{1-q}{r}\right)^3 \frac{(n-3)\cdots(m+n-5)}{1.2.3\cdots(m-1)} \\
& + \frac{q^{n-3}p^{m-2}}{r^{m+n-5}}\,{}_2K_3\,\frac{(n-2)\cdots(n+m-5)}{1.2.3\cdots(m-2)} \\
& + \frac{q^{n-2}p^{m-3}}{r^{m+n-5}}\,{}_3K_2\,\frac{(n-1)\cdots(n+m-5)}{1.2.3\cdots(m-3)} \\
& + \frac{q^{n-1}p^{m-4}}{r^{m+n-5}}\left(\frac{1-p}{r}\right)^3 \frac{n\cdots(n+m-5)}{1.2.3\cdots(m-4)}.
\end{aligned}
$$

We will determine ${}_2K_3$ and ${}_3K_2$ by means of the following equations:

$$
\begin{aligned}
r^3({}_2K_3 + 3\,{}_2L_3 + 3\,{}_2M_3 + {}_2N_3) = 1, \\
r^3({}_3K_2 + 3\,{}_3L_2 + 3\,{}_3M_2 + {}_3N_2) = 1.
\end{aligned}
$$

The law of the other coefficients ${}_mI_n$, ${}_mH_n, \ldots$ is clear, and it is easy, consequently, to determine them. As for the coefficient ${}_mC_n$, we will determine it by this equation

$$
1 = r^{m+n-2}\left[{}_mC_n + (m+n-2)\,{}_mD_n + \frac{(m+n-2)(m+n-3)}{1.2}\,{}_mE_n + \cdots\right].
$$

Thus we have therefore a general expression for ${}_{m,n}y_x$ and, consequently, the probability of player C winning; by the same method, and by means of analogous formulas, we would have that of the two other players A and B; in such a way that we have a solution of the Problem of points in the case of three players; a Problem which had not yet been solved, as I know, although the geometers who have occupied themselves in the analysis of chances seemed to desire the solution. (See *Mr. Montmort*, in his work *Sur l'analyse des jeux de hasard*, second edition, page 247.)

I assume in the expression ${}_{m,n}y_x$, $m = 2, n = 3$ and $x = 9$, that is to say that the number of trials which fall to player C is 4: I assume, moreover, $p = q = r = \frac{1}{3}$. This put, we will have

$${}_{2,3}y_x = \frac{x-3}{3^{x-2}}\left(\frac{xx+2}{2}\right),$$

and, by supposing $x = 9$, we will have the probability of C, for winning, equal to ${}_{2,3}y_9 = \frac{83}{729}$; in order to have the probability of B, I observe that it is equal to ${}_{2,4}y_9$; now we have

$${}_{2,4}y_x = \frac{1}{3^{x-2}}\left[4\frac{(x-2)(x-3)(x-4)(x-5)}{1.2.3.4} + 8\frac{(x-2)(x-3)(x-4)}{1.2.3} + 7\frac{(x-2)(x-3)}{1.2} + 5(x-2) - 17\right]$$

If we suppose $x = 9$, we will have

$${}_{2,4}y_9 = \frac{195}{729};$$

the probability of A equals $1 - \frac{83}{729} - \frac{195}{729} = \frac{451}{729}$.

The preceding method could take place again, if, instead of three players, one supposed a greater number.

One can solve the preceding Problem by the method of combinations in an extremely simple manner that is here:

The same things being assumed as in the preceding Problem; let, moreover, i be the number of trials which lacks to player C, so that we have $x = m+n+i$; it is evident that the game must end at the latest in $x-2$ trials; therefore the number of all the possible cases, multiplied each by their particular probability, is $(p+q+r)^{m+n+i-2}$. In order to have the number of all the cases in which the player A wins, it is necessary to develop the trinomial $(p+q+r)^{m+n+i-2}$ and to admit only the terms in which p has an exponent equal or superior to m; let therefore $Hp^{m+\mu}q^{\nu}r^{n+i-2-\mu-\nu}$ be one of the terms; if the exponents of q and of r are one less than n, and the other less than i, it is necessary to admit this term in whole; but, if the exponent of q, for example, is equal or greater than n, it is necessary to reject from this term all the combinations in which q happens n times before p happens m times. Let therefore $\nu = n+\lambda$; I observe, this put, that these combinations are: 1° those in which, p having happened $m-1$ times, q has happened precisely n times; 2° those in which, p having happened $m-2$ times, q has happened precisely $n+1$ times; 3° those in which, p having happened $m-3$ times, q has happened precisely $n+2$ times, etc., and thus in sequence until the combination in which, p having happened $m-\lambda-1$ times, q has happened $n+\lambda$ times, if however λ does not exceed $m-1$; because, otherwise, it would be necessary to stop at the combination in which p does not happen at all; presently, the number of cases in which, out of $m+n-1$ trials, p will happen $m-1$, and q, n times, is, as one knows,

$$\frac{\Delta(m+n-1)}{\Delta(n)\Delta(m-1)};$$

but, as in the term $Hp^{m+\mu}q^{n+\lambda}r^{i-2-\mu-\lambda}$, p happens $m+\mu$ times, and q, $n+\lambda$ times, it is necessary to multiply $\frac{\Delta(m+n-1)}{\Delta(n)\Delta(m-1)}$ by the number of combinations in which, p happening $\mu+1$ times, q happens λ times; now the number of these combinations is

$$\frac{\Delta(\mu+\lambda+1)}{\Delta(\mu+1)\Delta(\lambda)};$$

therefore we will have

$$\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n)\Delta(\lambda)\Delta(m-1)\Delta(\mu+1)}$$

for the number of combinations in which q has happened n times, when p has yet happened only $m-1$ times; we will find similarly

$$\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n+1)\Delta(\lambda-1)\Delta(m-2)\Delta(\mu+2)}$$

for the number of cases in which q has happened $n+1$ times, when p has not yet happened $m-2$ times, and thus in sequence. Let therefore

$$Q_{\mu+\lambda}=\left[1+\frac{\lambda(m-1)}{(n-1)(\mu+2)}+\frac{\lambda(\lambda-1)(m-1)(m-2)}{(n+1)(n+2)(\mu+2)(\mu+3)}+\cdots\right]$$
$$\times\frac{\Delta(m+n-1)\Delta(\mu+\lambda+1)}{\Delta(n)\Delta(m-1)\Delta(\mu+1)\Delta(\lambda)}p^{m+\mu}q^{n+\lambda}r^{i-2-\mu-\lambda};$$

let us designate as $(Q_{\mu+\lambda})$ the sum of all the terms which one can form, by giving to μ and to λ, in $Q_{\mu+\lambda}$, all the possible values in whole and positive numbers from zero, in a manner however that $\mu+\lambda$ never exceed $i-2$; let us express next by $(R_{\mu+\lambda})$ that which $(Q_{\mu+\lambda})$ becomes, when we change q to r, n to i, and reciprocally; this put, the probability of A, for winning, will be

$$\frac{1}{(p+q+r)^{m+n+i-2}}=\left[p^{m+n+i-2}+\frac{m+n+i-2}{1}p^{m+n+i-3}(q+r)+\cdots\right.$$
$$\left.+\frac{(m+n+i-2)\cdots(m+i-1)}{1.2.3\cdots(n-2)}p^{m}(q+r)^{n+i-2}-(Q_{\mu+\lambda})-(R_{\mu+\lambda})\right].$$

The same method has equal place, whatever be the number of players.

XXXII.

PROBLEM XVI. — *I suppose the tickets* A1, A2, B1 *and* B2*, contained in an urn, and that two players* A *and* B *play on this condition that* A *choosing the tickets* A1 *and* A2*, and* B *the two others, if one draws each time one alone of these tickets at random, the one of the two players will win, who first will have attained the number i, the tickets* A1 *and* B1 *counting for* 1*, and the tickets* A2 *and* B2 *counting for* 2*. This put, if there lacks n units to the player* A*, and $x-n$ units to player* B*, one asks the respective probabilities of the two players* A *and* B *to win.*

Let ${}_n y_x$ be the probability of B winning; if one draws from the urn the ticket A1, it will become ${}_{n-1} y_{x-1}$; if one draws the ticket A2, it will become ${}_{n-2} y_{x-2}$; if the ticket B1 comes out, it will be ${}_n y_{x-1}$; if it is the ticket B2, it will be ${}_n y_{x-2}$; we will have therefore

$$(1) \qquad {}_n y_x = \frac{1}{4}\,{}_n y_{x-1} + \frac{1}{4}\,{}_n y_{x-2} + \frac{1}{4}\,{}_{n-1} y_{x-1} + \frac{1}{4}\,{}_{n-2} y_{x-2}.$$

This equation is integrated as in Problem VII; but, for this, it is necessary to have two particular equations in the two particular suppositions for n. Now, if one supposes $n = 0$, we have ${}_0 y_x = 0$, and if one supposes $n = 1$, ${}_1 y_x = \frac{1}{2}\,{}_1 y_{x-1}$, because I suppose that then the two players exclude the tickets A2 and B2. We have therefore, by Problem VII,

$${}_n y_x = a_n \cdot {}_n y_{x-1} + {}^1 a_n \cdot {}_n y_{x-2} + {}^2 a_n \cdot {}_n y_{x-3} + \cdots,$$

and the equation

$$1 = \frac{a_n}{f} + \frac{{}^1 a_n}{f^2} + \frac{{}^2 a_n}{f^3} + \cdots$$

is the same as this

$$0 = \left(1 - \frac{1}{2f}\right)\left(1 - \frac{1}{4f} - \frac{1}{4ff}\right)^{n-1};$$

we will have thus

$$\begin{aligned} {}_n y_x = \frac{A_n}{2^x} + p^x \Bigg[& N_n \frac{x(x-1)\cdots(x-n+3)}{1.2.3\ldots(n-2)} + M_n \frac{x(x-1)\cdots(x-n+4)}{1.2.3\ldots(n-3)} \\ & + L_n \frac{x(x-1)\cdots(x-n+5)}{1.2.3\ldots(n-4)} + K_n \frac{x(x-1)\cdots(x-n+6)}{1.2.3\ldots(n-5)} \\ & + \cdots + C_n \\ + {}^1 p^x \Bigg[& {}^1 N_n \frac{x(x-1)\cdots(x-n+3)}{1.2.3\ldots(n-2)} + \cdots \Bigg], \end{aligned}$$

p and ${}^1 p$ being the two roots of the equation

$$f^2 - \frac{1}{4} f = \frac{1}{4},$$

that is p being $\frac{1+\sqrt{17}}{8}$, and ${}^1 p$ being $\frac{1-\sqrt{17}}{8}$.

It is necessary now to determine the arbitrary constants $A_n, N_n, \ldots$. Now, if one substitutes into equation (1), in place of ${}_n y_x$, ${}_n y_{x-1}$, ${}_{n-1} y_{x-1}$, … their values drawn from the expression of ${}_{n}y_{x}$, we will have

$$\begin{aligned}
&\frac{A_n}{2^x}+p^x\Bigg[N_n\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+(2N_n+M_n)\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)}\\
&\qquad+(N_n+2M_n+L_n)\frac{(x-2)\cdots(x-n+3)}{1.2.3\ldots(n-4)}\\
&\qquad+(M_n+2L_n+K_n)\frac{(x-2)\ldots(x-n+4)}{1.2.3\ldots(n-5)}\\
&\qquad\qquad+\cdots C_n]\\
&+{}^1p^x\left[{}^1N_n\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+\cdots\right]
\end{aligned}$$

$$\begin{aligned}
=\frac{1}{4}p^x\Bigg\{&N_n\left(\frac{1}{p}+\frac{1}{p^2}\right)\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}\\
&+\left[\frac{N_n}{p}+M_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{N_{n-1}}{p}\right]\frac{(x-2)\cdots(x-n+2)}{1.2.3\ldots(n-3)}\\
&+\left[\frac{M_n}{p}+L_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{M_{n-1}}{p}+\frac{N_{n-1}}{p}+\frac{N_{n-2}}{p^2}\right]\frac{(x-2)\cdots(x-n+3)}{1.2.3\ldots(n-4)}\\
&\quad+\left[\frac{L_n}{p}+K_n\left(\frac{1}{p}+\frac{1}{p^2}\right)+\frac{L_{n-1}}{p}+\frac{M_{n-1}}{p}+\frac{M_{n-2}}{p^2}\right]\frac{(x-2)\cdots(x-n+4)}{1.2.3\ldots(n-5)}+\cdots\Bigg\}\\
&+\frac{1}{4}\,{}^1p^x\left\{{}^1N_n\left(\frac{1}{{}^1p}+\frac{1}{{}^1p^2}\right)\frac{(x-2)\cdots(x-n+1)}{1.2.3\ldots(n-2)}+\cdots\right\}\\
&+\frac{1}{4}\frac{A_n}{2^{x-1}}+\frac{1}{4}\frac{A_n}{2^{x-2}}+\frac{1}{4}\frac{A_{n-1}}{2^{x-1}}+\frac{1}{4}\frac{A_{n-2}}{2^{x-2}}.
\end{aligned}$$

Whence, by considering that

$$1=\frac{1}{4p}+\frac{1}{4pp},$$

we will form the following equations:

$$\begin{aligned}
0&=\frac{1}{2}A_n+\frac{1}{2}A_{n-1}+A_{n-2},\\
2N_n&=\frac{1}{4}\frac{N_n}{p}+\frac{1}{4}\frac{N_{n-1}}{p},\\
2M_n+N_n&=\frac{1}{4}\frac{M_n}{p}+\frac{1}{4}\frac{M_{n-1}}{p}+\frac{1}{4}\frac{N_{n-2}}{p^2}+\frac{1}{4}\frac{N_{n-1}}{p},\\
2L_n+M_n&=\frac{1}{4}\frac{L_n}{p}+\frac{1}{4}\frac{L_{n-1}}{p}+\frac{1}{4}\frac{M_{n-2}}{p^2}+\frac{1}{4}\frac{M_{n-1}}{p},\\
&\vdots
\end{aligned}$$

We will have some similar equations for ${}^1N_n$, ${}^1M_n,\ldots$ We will determine the quantities C_n and ${}^1C_n$, by considering that, when $n=x$, ${}_{n}y_{x}=1$, and that, when $x=2n$,

${}_{n}y_{x} = \frac{1}{2}$; whence we obtain the equations

$$1 = \frac{A_n}{2^n} + p^n \left[C_n + nD_n + \cdots + \frac{n(n-1)\cdots 3}{1.2.3\ldots(n-2)} N_n \right] + {}^1p^n \left[{}^1C_n + n\,{}^1D_n + \cdots \right]$$

and

$$\frac{1}{2} = \frac{A_n}{2^{2n}} + p^{2n} \left[C_n + 2nD_n + \cdots + N_n \frac{2n\cdots(n+3)}{1.2.3\ldots(n-2)} \right] + {}^1p^{2n} \left[{}^1C_n + 2n\,{}^1D_n + \cdots + {}^1N_n \frac{2n\cdots(n+3)}{1.2\ldots(n-2)} \right].$$

It is necessary now to integrate the preceding equations. Now, if one makes $-\frac{1}{2\sqrt{2}} = \cos q$ and $\frac{\sqrt{7}}{2\sqrt{2}} = \sin q$, which gives very nearly $q = 110^\circ\,42'$, we will find (article IX)

$$A_n = 2^{\frac{n}{2}}(\alpha \cos nq + \beta \sin nq),$$

α and β being two arbitrary constants. Now, if one makes $n = 0$, we have

$$A_0 = 0 = \alpha;$$

and if one makes $n = 1$, we have

$$A_n = \frac{1}{2},$$

because ${}_{1}y_{x} = \frac{1}{2^{x-1}}$; therefore

$$\beta\sqrt{2}\sin q = \frac{1}{2} \quad \text{and} \quad \beta = \frac{1}{2\sqrt{2}\sin q};$$

hence

$$A_n = 2^{\frac{n-2}{2}} \frac{\sin nq}{\sin q}.$$

The equation

$$2N_n = \frac{1}{4}\frac{N_n}{p} + \frac{1}{4}\frac{N_{n-1}}{p}$$

gives

$$N_n = \frac{Q}{(8p-1)^{n-2}}.$$

This value of N_n commences to take place only when $n = 2$; therefore

$$Q = N_2 \quad \text{and} \quad N_n = \frac{N_2}{(8p-1)^{n-2}};$$

similarly

$${}^1N_n = \frac{{}^1N_2}{(8\,{}^1p-1)^{n-2}}.$$

We will determine N_2 and ${}^1N_2$ by these equations

$$1 = \frac{A_2}{2^2} + p^2 \cdot N_2 + {}^1p^2 \cdot {}^1N_2$$
$$\frac{1}{2} = \frac{A_2}{2^4} + p^4 \cdot N_2 + {}^1p^4 \cdot {}^1N_2$$

We will determine in the same manner the other coefficients $M_n, L_n, K_n, \ldots$.

XXXIII.

PROBLEM XVII. — *Two players* A *and* B *play to this condition, that at each trial, the one who loses will give an écu to the other; I suppose that the skill of* A *be to that of* B*, as p is to q, and that both have a number m of écus; we ask what is the probability that the game will end before, or at the number x of trials.*

I suppose first $p = q$. Let

${}_0y_x$ be the number of cases according to which, at trial x, the gain of the two players is null;

${}_1y_x$ be the number of cases according to which the gain of one or the other is 1;

${}_2y_x$ be the number of cases following which the gain is 2, and thus in sequence. This put, we will form the following equations:

$$(\psi)\quad \begin{cases} {}_0y_x = {}_1y_{x-1}, \\ {}_1y_x = 2 \cdot {}_0y_{x-1} + {}_2y_{x-1}, \\ {}_2y_x = {}_1y_{x-1} + {}_3y_{x-1}, \\ {}_3y_x = {}_2y_{x-1} + {}_4y_{x-1}, \\ \vdots \\ (\sigma)\quad {}_ny_x = {}_{n-1}y_{x-1} + {}_{n+1}y_{x-1}, \\ \vdots \\ {}_{m-1}y_x = {}_{m-2}y_{x-1} \end{cases}$$

In order to show by what process one obtains these equations, I observe that, at each trial, there can happen two different cases, namely, that A wins, or that it is B; now it is clear that the gain cannot be zero at the trial x, without having been 1 at the trial $x-1$, and each case in which it is 1 at trial $x-1$ gives a case in which it is null at trial x; whence I deduce the equation

$${}_0y_x = {}_1y_{x-1}.$$

Next all the cases in which the gain is null at trial $x-1$ each give two cases in which there is 1 at trial x; whence we will have

$${}_1y_x = 2 \cdot {}_0y_{x-1} + {}_2y_{x-1}.$$

It is likewise in the other equations. Finally, we will obtain the last by considering that one must exclude the term ${}_{m}y_{x-1}$, because this term cannot take place, as long as the game is supposed not finite.

The number of all possible cases is 2^x; because, by naming h_x this number, as there can happen at the following trial two different cases, namely, that A beats B or that B beats A, the number h_x, being able to be combined with these two cases, gives consequently $2h_x$ for the number of all possible cases at trial $x+1$; we have therefore

$$h_{x+1} = 2h_x;$$

whence, by integrating,

$$h_x = A2^x,$$

A being an arbitrary constant. Now, putting $x = 1$, $h_x = 2$; therefore

$$A = 1 \quad \text{and} \quad h_x = 2^x.$$

Let presently u_x be the probability that the game will end precisely at the number x of trials: we will have

$$u_x = \frac{{}_{m}y_{x}}{2^x};$$

but we have clearly

$${}_{m}y_{x} = {}_{m-1}y_{x-1};$$

therefore

$$u_x = \frac{{}_{m-1}y_{x-1}}{2^x}.$$

Let z_x be the probability that the game will end before or at the number x of trials, we will have

$$z_x = z_{x-1} + u_x;$$

therefore

$$\Delta z_{x-1} = \frac{{}_{m-1}y_{x-1}}{2^x} \quad \text{or} \quad 2^{x+1}\Delta z_x = {}_{m-1}y_{x}.$$

There is therefore no more but to determine the value of ${}_{m-1}y_{x}$, which can be made by means of the preceding equations (ψ). For this, I observe that these equations are able to correspond to Problem VIII by means of a simple preparation; now this preparation consists to form, by means of the first two, an equation among three variables, which we will make by substituting into the second, in place of ${}_{0}y_{x-1}$, its value ${}_{1}y_{x-2}$ deduced from the first, and we will have

$${}_{1}y_{x} = 2 \cdot {}_{1}y_{x-2} + {}_{2}y_{x-1}.$$

Let now

$$(\Omega) \quad {}_{n}y_{x} = a_n \cdot {}_{n}y_{x-2} + {}^{1}a_n \cdot {}_{n}y_{x-4} + \cdots + u_n + b_n \cdot {}_{n+1}y_{x-1} + {}^{1}b_n \cdot {}_{n+1}y_{x-3} + \cdots,$$

It is not necessary to take account, in this equation, of the terms ${}_{n}y_{x-1}$, ${}_{n}y_{x-3}$, $\ldots$, ${}_{n+1}y_{x-2}$, ${}_{n+1}y_{x-4}$, $\ldots$, because these terms are null as soon as ${}_{n}y_{x}$ has any value, seeing that, if the gain is even or odd at trial x, it is necessarily odd or even at the trials $x-1, x-3, \ldots$. This put, the equation (Ω) gives

$$
\begin{aligned}
{}_{n-1}y_{x-1} = a_{n-1} \cdot {}_{n-1}y_{x-3} + {}^1a_{n-1} \cdot {}_{n-1}y_{x-5} + \cdots + u_{n-1} \\
+ b_{n-1} \cdot {}_{n}y_{x-2} + {}^1b_{n-1} \cdot {}_{n}y_{x-4} + \cdots
\end{aligned}
$$

If one substitutes into this equation, in place of ${}_{n-1}y_{x-1}$, ${}_{n-1}y_{x-3}, \ldots$, their values that equation (σ) gives, we will have, after having ordered,

$$
\begin{aligned}
{}_{n}y_{x} = (a_{n-1} + b_{n-1})\,{}_{n}y_{x-2} + ({}^1a_{n-1} + {}^1b_{n-1})\,{}_{n}y_{x-4} + ({}^2a_{n-1} + {}^2b_{n-1})\,{}_{n}y_{x-6} + \cdots \\
+ {}_{n+1}y_{x+1} - a_{n-1} \cdot {}_{n+1}y_{x-3} - {}^1a_{n-1} \cdot {}_{n+1}y_{x-5} - \cdots + u_{n-1}.
\end{aligned}
$$

By comparing this equation with equation (Ω), we will have

$$
\begin{aligned}
b_n &= 1, \\
a_n &= a_{n-1} + b_{n-1}, \\
{}^1b_n &= -a_{n-1}, \\
{}^1a_n &= {}^1a_{n-1} + {}^1b_{n-1}, \\
{}^2b_n &= -{}^1a_{n-1}, \\
{}^2a_n &= {}^2a_{n-1} + {}^2b_{n-1}, \\
&\vdots \\
u_n &= u_{n-1}.
\end{aligned}
$$

In order to integrate these equations, it is necessary to make the following considerations:

The first equation begins to take place when $n = 1$.

The second begins to exist only when $n = 2$; thus, the arbitrary constant which comes by integrating must be determined by means of the value of a_n when $n = 1$.

The third equation begins to exist when $n = 2$.

The fourth begins to exist only when $n = 3$; and the arbitrary constant which comes by integrating must be determined by means of the value of ${}^1a_n$, when $n = 2$; and thus for the rest.

This put, if one integrates the second equation, we will have

$$a_n = n + C,$$

C being an arbitrary constant; now, putting $n = 1$, we have

$$a_n = 2, \quad \text{thus} \quad C = 1;$$

hence

$${}^1b_n = -a_{n-1} = -n.$$

One must observe that this equation begins to exist only when $n = 2$; now, n being 1, we have

$${}^1b_1 = 0, \quad {}^2b_1 = 0, \quad \ldots,$$

moreover, by making $n=2$, we have

$$^{2}b_2 = -\,^{1}a_1 = 0;$$

likewise,

$$^{3}b_2 = 0, \quad ^{4}b_2 = 0, \quad \ldots, \quad ^{1}a_2 = \,^{1}a_1 + \,^{1}b_1 = 0;$$

similarly,

$$^{2}a_2 = 0, \quad ^{3}a_2 = 0, \quad \ldots,$$

If one integrates the fourth equation, we will have

$$^{1}a_n = -\frac{(n+1)(n-2)}{1.2} + C;$$

in order to determine the constant C, one avails oneself of the value of $^{1}a_2$; we have

$$^{1}a_2 = 0, \quad \text{therefore} \quad C = 0;$$

hence

$$^{2}b_n = \frac{n(n-3)}{1.2};$$

this expression of $^{2}b_n$ is able to begin to take place, by the remarks preceding, only when $n=3$; moreover, by making $n=3$, we have

$$^{3}b_3 = -\,^{2}a_2 = 0;$$

similarly,

$$^{4}b_3 = 0, \quad ^{5}b_3 = 0, \quad \ldots, \quad ^{2}a_3 = \,^{2}a_2 + \,^{2}b_2 = 0;$$

similarly,

$$^{3}a_3 = 0, \quad ^{4}a_3 = 0, \quad \ldots$$

The sixth equation gives, by integrating,

$$^{2}a_n = \frac{(n+1)(n-3)(n-4)}{1.2.3} + C.$$

In order to determine C, I observe that $^{2}a_3$ equals 0; therefore, $C=0$. Hence

$$^{2}b_n = -\frac{n(n-4)(n-5)}{1.2.3},$$

an expression which is able to begin to exist only when $n=4$, and thus in sequence.

Finally, $u_n = u_{n-1}$; therefore, $u_n = C$. Now, putting $n=1$, $u_n = 0$; therefore, $C=0$. Thus we will have

$$\begin{aligned} {}_{n}y_{x} =& (n+1)\,{}_{n}y_{x-2} - \frac{(n+1)(n-2)}{1.2}\,{}_{n}y_{x-4} \\ &+ \frac{(n+1)(n-3)(n-4)}{1.2.3}\,{}_{n}y_{x-6} - \cdots \\ &+ {}_{n+1}y_{x-1} - n\cdot{}_{n+1}y_{x-3} + \frac{n(n-3)}{1.2}\,{}_{n+1}y_{x-5} - \cdots \end{aligned}$$

If one supposes now $n=m-1$, then it is not necessary to take account of the terms ${}_{n+1}y_{x-1}$, ${}_{n+1}y_{x-3}$, ... because these terms are excluded from the equations (ψ); we will have therefore

$$ {}_{m-1}y_x = m\cdot {}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2}\,{}_{m-1}y_{x-4} + \frac{m(m-4)(m-5)}{1.2.3}\,{}_{m-1}y_{x-6} - \cdots $$

If one substitutes presently into this equation, in place of ${}_{m-1}y_x$, its value $2^{x+1}\Delta z_x$, we will have, after having integrated,

$$ z_x = m\frac{1}{2^2}z_{x-2} - \frac{m(m-3)}{1.2}\frac{1}{2^4}z_{x-4} + \frac{m(m-4)(m-5)}{1.2.3}\frac{1}{2^6}z_{x-6} + \cdots + C. $$

I suppose now the skills of two players unequal in the ratio of p to q; let $p+q=1$. This put, if one asks for the probability of the following combination

$$ \begin{array}{cccccccccc} 1, & 2, & 3, & 4, & 5, & 6, & 7, & \ldots, & x, \\ p, & q, & q, & p, & p, & p, & q, & \ldots, & q, \end{array} $$

which signifies A wins on the first trial, B on the second and on the third, A on the fourth, fifth, and sixth, etc. It is clear that, in order to have this probability, one must multiply all these quantities by one another; naming therefore r the number of times that p is found repeated in this combination, $x-r$ will express how many times q is found repeated; the probability of this combination will be consequently $p^r q^{x-r}$.

If one makes $x-r=r+s$, and if in some place one stops the combination, the number of times that one of the quantities p and q is found more often repeated than the other is always less than m, this combination will be one of those in which B will gain s écus to player A; now, one is able to make a corresponding combination in which A will gain s écus to B, and the probability of this combination will be $q^r p^{r+s}$, the ratio of this probability to the preceding is that of p^s to q^s; whence there results that generally the number of cases according to which A gains s écus to B, each multiplied by their particular probability, is to the number of cases according to which B gains s écus to player A, multiplied by their probability, as $p^s : q^s$.

This put, let ${}_0y_x$ be the number of cases according to which at trial x the gain of the two players is null, each multiplied by their probability. Let ${}_1y_x$, ${}_2y_x$, ... be the number of cases according to which the gain of player A is $1, 2, \ldots$ écus, each multiplied by their particular probability, and if ${}_1\overset{1}{y}_x$, ${}_2\overset{1}{y}_x$, ... express the analogous quantities for player B; it is easy, now by some considerations entirely similar to those according to which I have formed the equations (ψ), to obtain the following:

$$ (\psi') \qquad \left\{ \begin{array}{ll} & {}_0y_x = q\cdot {}_1y_{x-1} + p\cdot {}_1\overset{1}{y}_{x-1}, \\ & {}_1y_x = p\cdot {}_0y_{x-1} + q\cdot {}_2y_{x-1}, \\ & {}_2y_x = p\cdot {}_1y_{x-1} + q\cdot {}_3y_{x-1}, \\ & \vdots \\ (\sigma') & {}_ny_x = p\cdot {}_{n-1}y_{x-1} + q\cdot {}_{n+1}y_{x-1}, \\ & \vdots \\ & {}_{m-1}y_x = p\cdot {}_{m-2}y_{x-1} \end{array} \right. $$

Now we have, by the preceding remarks,

$$p \cdot {}_1\overset{1}{y}_{x-1} = q \cdot {}_1y_{x-1}.$$

The first equation becomes therefore

$$_0y_x = 2q \cdot {}_1y_{x-1},$$

hence

$$_0y_{x-1} = 2q \cdot {}_1y_{x-2};$$

substituting this value of $_0y_{x-1}$ into the second, we will have

$$_1y_x = 2qp \cdot {}_1y_{x-1} + q \cdot {}_2y_{x-1};$$

it is easy to see that the equations (ψ') correspond in this way to Problem VIII. Let there be therefore

$$_ny_x = a_n \cdot {}_ny_{x-2} + {}^1a_n \cdot {}_ny_{x-4} + \cdots + u_n + b_n \cdot {}_{n+1}y_{x-1} + {}^1b_n \cdot {}_{n+1}y_{x-3} + \cdots,$$

and we will find, by operating exactly as I have done above, when p and q were equal,

$$\begin{aligned} _ny_x = (n+1)pq \cdot {}_ny_{x-2} - \frac{(n+1)(n-2)}{1.2}p^2q^2 \cdot {}_ny_{x-4} + \cdots \\ + q \cdot {}_{n+1}y_{x-1} - npq^2 \cdot {}_{n+1}y_{x-3} + \cdots \end{aligned}$$

Therefore, if one supposes $n = m-1$, we will have

$$(\varpi) \qquad _{m-1}y_x = mpq \cdot {}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2}p^2q^2 \cdot {}_{m-1}y_{x-4} + \cdots;$$

by rejecting the terms $_my_{x-1}$, $_my_{x-3}$, ... which can have no place, according to the supposition that the game does not end before the trial x. Let now u_x be the probability that the game will end precisely at trial x, it is clear that we will have

$$u_x = {}_my_x + {}_m\overset{1}{y}_x;$$

now we have $_my_x : {}_m\overset{1}{y}_x :: p^m : q^m$; therefore

$$u_x = \left(1 + \frac{q^m}{p^m}\right) {}_my_x;$$

moreover,

$$_my_x = p \cdot {}_{m-1}y_{x-1};$$

hence,

$$u_x = p\left(1 + \frac{q^m}{p^m}\right) {}_{m-1}y_{x-1}.$$

Let z_x be the probability that the game will end before or at trial x, we will have

$$\Delta z_x = u_{x+1} = p\left(1+\frac{q^m}{p^m}\right)\,{}_{m-1}y_x;$$

by substituting therefore, in place of ${}_{m-1}y_x$ this value in equation (ϖ), we will have, after having integrated,

$$(\varpi) \qquad \begin{cases} z_x = mpqz_{x-2} - \dfrac{m(m-3)}{1.2}p^2q^2z_{x-4} \\ \qquad\qquad + \dfrac{m(m-3)(m-5)}{1.2.3}p^3q^3z_{x-6} - \cdots + C. \end{cases}$$

In order to determine the arbitrary constant C, I observe that, as long as x is less than m, z_x equals 0, and that x being equal to m, z_x equals p^m+q^m; therefore,

$$C = p^m + q^m.$$

Let $1-t_x = z_x$; t_x will express consequently the probability that the game will not end before or at trial x, and we will have

$$\begin{aligned} t_x = mpqt_{x-2} - \frac{m(m-3)}{1.2}p^2q^2t_{x-4} + \cdots \\ - p^m - q^m + \left[1 - mpq + \frac{m(m-3)}{1.2}p^2q^2 - \cdots\right]. \end{aligned}$$

Now it is remarkable that we have, whatever be m, and by supposing $p+q=1$,

$$0 = 1 - p^m - q^m - mpq + \frac{m(m-3)}{1.2}p^2q^2 - \cdots,$$

or, generally, by supposing any p and q,

$$(p+q)^m = mpq(p+q)^{m-2} - \frac{m(m-3)}{1.2}p^2q^2(p+q)^{m-4} + \cdots + p^m + q^m;$$

it is this of which would be able to be convinced by induction, by giving to m different numerical values, but here is a general demonstration of it. We have

$$\begin{aligned} p+q &= p+q, \\ (p+q)^2 &= 2pq(p+q)^0 + p^2 + q^2, \\ (p+q)^3 &= 3pq(p+q) + p^3 + q^3, \\ &\vdots \end{aligned}$$

Let therefore, in general,

$$(\tau) \qquad (p+q)^m = A_m(p+q)^{m-2} + {}^1A_m(p+q)^{m-4} + \cdots + p^m + q^m,$$

and we will have

$$\begin{aligned}(p+q)^{m+1} =& A_m(p+q)^{m-1} + {}^1A_m(p+q)^{m-3} + \cdots \\ & + p^{m+1} + q^{m+1} + pq(p^{m-1} + q^{m-1}).\end{aligned}$$

Now we have

$$p^{m-1} + q^{m-1} = (p+q)^{m-1} - A_{m-1}(p+q)^{m-3} - \cdots;$$

therefore

$$\begin{aligned}(p+q)^{m+1} =& (A_m + pq)(p+q)^{m-1} \\ & + ({}^1A_m - A_{m-1}pq)(p+q)^{m-3} + \cdots + p^{m+1} + q^{m+1}.\end{aligned}$$

We have moreover

$$(p+q)^{m+1} = A_{m+1}(p+q)^{m-1} + {}^1A_{m+1}(p+q)^{m-3} + \cdots + p^{m+1} + q^{m+1};$$

whence, by comparing, we will have

$$\begin{aligned}A_{m+1} &= A_m + pq, \\ {}^1A_{m+1} &= {}^1A_m - A_{m-1}pq, \\ {}^2A_{m+1} &= {}^2A_m - {}^1A_{m-1}pq, \\ &\vdots\end{aligned}$$

All these equations are not able to exist at once; the first begins to take place only when $m = 1$; the second, when $m = 2$; the third, when $m = 3$; etc. Moreover, as they assume necessarily known the expressions of $p+q$ and $(p+q)^2$, in order to determine next, in their way, $(p+q)^3$, $(p+q)^4, \ldots$, there results that the law represented by these equations begins to take place when $m+1 = 3$; thus, the first equation begins to exist when $m = 2$; the second, when $m = 3$; the third, when $m = 4$, etc.

This put, by integrating the first, we have

$$A_m = mpq + C.$$

Now, putting $m = 2$, we have

$$A_2 = 2pq;$$

therefore, $C = 0$.

Next, the second gives

$${}^1A_m = -\frac{m(m-3)}{1.2}p^2q^2 + C;$$

now, putting $m = 3$, ${}^1A_3 = 0$, because $(p+q)$ is not able to have negative exponent in the formula (τ); therefore $C = 0$, and thus for the rest. Therefore

$$(p+q)^m = mpq(p+q)^{m-2} - \frac{m(m-3)}{1.2}p^2q^2(p+q)^{m-4} + \cdots + p^m + q^m;$$

thus we will have

$$(\delta)\qquad t_{x} = mpqt_{x-2} - \frac{m(m-3)}{1.2}p^2q^2t_{x-4} + \cdots$$

In order to integrate this equation, I begin by observing that it is differential of order $\frac{m}{2}$ or $\frac{m-1}{2}$, according as m is even or odd. Moreover, it is easy to see, by inspection of the equations (ψ'), that it begins to exist when $x = m$. Thus, the arbitrary constants which come by the integration must be determined by the values of t_x, when one makes $x = 0, x = 2, x = 4, \ldots, x = m-2$ or $x = 1, x = 3, x = 5, \ldots, x = m-2$, according as m is even or odd. Now, all these values are equal to unity, because it is certain that the game cannot end before m trials.

Presently, if one supposes x' equal to $\frac{x}{2}$ or $\frac{x-1}{2}$, according as m is even or odd, we will have

$$t_{x'} = mpqt_{x'-1} - \frac{m(m-3)}{1.2}p^2q^2t_{x'-2} + \cdots$$

The integral of this equation depends on the resolution of this algebraic equation

$$f^{\frac{m}{2}} = mpqf^{\frac{m}{2}-1} - \frac{m(m-3)}{1.2}p^2q^2f^{\frac{m}{2}-2} + \cdots,$$

if m is even, or of this

$$f^{\frac{me-1}{2}} = mpqf^{\frac{m-1}{2}-1} - \frac{m(m-3)}{1.2}p^2q^2f^{\frac{m-1}{2}-2} + \cdots,$$

if m is odd.

Now, if one makes $\cos\phi = y$, we have, as one knows,

$$\cos m\phi = 2^{m-1}y^m - m2^{m-3}y^{m-2} + \frac{m(m-3)}{1.2}2^{m-5}y^{m-4} - \cdots$$

Let $\cos m\phi = 0$, and we will have

$$0 = y^m - m\frac{1}{4}y^{m-2} + \frac{m(m-3)}{1.2}\frac{1}{4^2}y^{m-4} - \cdots$$

when m is even, or

$$0 = y^{m-1} - m\frac{1}{4}y^{m-3} + \frac{m(m-3)}{1.2}\frac{1}{4^2}y^{m-5} - \cdots$$

when m is odd.

The different values of y in this equation are the cosines of the different arcs, which, multiplied by m, have their cosines equal to zero; now the arcs which have their cosines null are $\frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \pi$ expressing the semi-circumference of which the radius is unity. The different values of y are, consequently, plus and minus the cosines of the arcs $\frac{\pi}{2m}, \frac{3\pi}{2m}, \frac{5\pi}{2m}, \ldots$ to $\frac{(m-1)\pi}{2m}$ or $\frac{(m-2)\pi}{2m}$ inclusively, according as m is even or odd; the cosines of the following arcs being the same, with the difference of signs excepted, the one of $\frac{\pi}{2}$ being null; let therefore $l, l_1, l_2, \ldots$ be these different cosines, the values of y

will be therefore $\pm l, \pm l_1, \ldots$. Now it is easy to see that $f = 4y^2pq$, hence, the different values of f will be $4l^2pq$, $4l_1^2pq$, ..., whence we will have

$$t_x = A(2l\sqrt{pq})^x + A_1(2l_1\sqrt{pq})^x + \cdots,$$

$A, A_1, \ldots$ being some arbitrary constants which will be determined by the method of article IX.

XXXIV.

PROBLEM XVIII. — *I have supposed, in the preceding problem, that the two players* A *and* B *had an equal number m écus; I suppose actually that player* A *has i écus, and player* B*, m écus; the rest subsisting, as above, we ask the probability that the game will end before, or at the number x of trials.*

It is easy to see that we will have first the equations (ψ') of the preceding Problem. Moreover, we will have the following:

$$(\psi'')\qquad \begin{cases} {}_1\overset{1}{y}_x = q\cdot {}_0y_{x-1} + p\cdot {}_2\overset{1}{y}_{x-1}, \\ {}_2\overset{1}{y}_x = q\cdot {}_1\overset{1}{y}_{x-1} + p\cdot {}_3\overset{1}{y}_{x-1}, \\ {}_3\overset{1}{y}_x = q\cdot {}_2\overset{1}{y}_{x-1} + p\cdot {}_4\overset{1}{y}_{x-1}, \\ \quad\vdots \\ {}_n\overset{1}{y}_x = q\cdot {}_{n-1}\overset{1}{y}_{x-1} + p\cdot {}_{n+1}\overset{1}{y}_{x-1}, \\ \quad\vdots \\ {}_{i-1}\overset{1}{y}_x = q\cdot {}_{i-2}\overset{1}{y}_{x-1}. \end{cases}$$

Let

$$\begin{array}{llll} {}_{i-1}\overset{1}{y}_x = {}_1\lambda_x & {}_{i-2}\overset{1}{y}_x = {}_2\lambda_x & {}_{i-3}\overset{1}{y}_x = {}_3\lambda_x & \ldots, \\ {}_0y_x = {}_i\lambda_x & {}_1y_x = {}_{i+1}\lambda_x & {}_2y_x = {}_{i+2}\lambda_x & \ldots, \end{array}$$

and we will have, by reuniting the equations (ψ') and (ψ''),

$$\begin{aligned} {}_1\lambda_x &= q\cdot {}_2\lambda_{x-1}, \\ {}_2\lambda_x &= q\cdot {}_3\lambda_{x-1} + p\cdot {}_1\lambda_{x-1}, \\ &\vdots \\ {}_{i+m-1}\lambda_x &= p\cdot {}_{i+m-2}\lambda_{x-1}. \end{aligned}$$

Let

$$(\Omega'')\quad \begin{cases} {}_n\lambda_{x-1} = a_n\cdot {}_n\lambda_{x-2} + {}^1a_n\cdot {}_n\lambda_{x-4} + {}^2a_n\cdot {}_n\lambda_{x-6} + \cdots + u_n \\ \qquad\qquad + b_n\cdot {}_{n+1}\lambda_{x-1} + {}^1b_n\cdot {}_{n+1}\lambda_{x-3} + {}^2b_n\cdot {}_{n-1}\lambda_{x-5} + \cdots, \end{cases}$$

and we will have

$$p\cdot {}_{n-1}\lambda_{x-1} = a_{n-1}p\cdot {}_{n-1}\lambda_{x-3} + {}^{1}a_{n-1}p\cdot {}_{n-1}\lambda_{x-5} + {}^{2}a_{n-1}p\cdot {}_{n-1}\lambda_{x-7} + \cdots + u_{n-1}p$$
$$+ b_{n-1}p\cdot {}_{n}\lambda_{x-2} + {}^{1}b_{n-1}p\cdot {}_{n}\lambda_{x-4} + \cdots$$

Now we have

$${}_{n}\lambda_{x} = q\cdot {}_{n+1}\lambda_{x-1} + p\cdot {}_{n-1}\lambda_{x-1};$$

therefore

$${}_{n}\lambda_{x} = (a_{n-1}+b_{n-1}p)\,{}_{n}\lambda_{x-2} + ({}^{1}a_{n-1}+{}^{1}b_{n-1}p)\,{}_{n}\lambda_{x-4} + ({}^{2}a_{n-1}+{}^{2}b_{n-1}p)\,{}_{n}\lambda_{x-6} + \cdots + u_{n-1}p$$
$$+ q\cdot {}_{n+1}\lambda_{x-1} - a_{n-1}q\cdot {}_{n+1}\lambda_{x-3} - {}^{1}a_{n-1}q\cdot {}_{n+1}\lambda_{x-5} - \cdots,$$

whence we will have, by comparing with equation (Ω''),

$$\begin{aligned}
b_n &= q,\\
a_n &= a_{n-1} + b_{n-1}p,\\
{}^{1}b_n &= -a_{n-1}q,\\
{}^{1}a_n &= {}^{1}a_{n-1} + {}^{1}b_{n-1}p,\\
{}^{2}b_n &= -{}^{1}a_{n-1}q,\\
{}^{2}a_n &= {}^{2}a_{n-1} + {}^{2}b_{n-1}p,\\
&\vdots\\
u_n &= u_{n-1}p.
\end{aligned}$$

One must observe that the first of these equations begins to exist when $n = 1$; the second and the third, when $n = 2$; the fourth and the fifth, when $n = 3$; etc.

This put, if one integrates the second, we will have

$$a_n = (n-1)pq + C;$$

now, putting $n = 1$, $a_n = 0$; thus $C = 0$, hence

$${}^{1}b_n = -a_{n-1}q = -(n-2)pq^2.$$

If we integrate the fourth, we will have

$${}^{1}a_n = -\frac{(n-2)(n-3)}{1.2}p^2q^2 + C;$$

in order to determine the constant C, I observe that, when $n = 2$, we have

$${}^{1}a_2 = {}^{1}a_1 + {}^{1}b_1 p = 0;$$

therefore $C = 0$, hence,

$${}^{2}b_2 = \frac{(n-3)(n-4)}{1.2}p^2q^3.$$

If we integrate the sixth equation, we will have

$$^2a_n = \frac{(n-3)(n-4)(n-5)}{1.2.3}p^3q^3 + C;$$

now we have

$$^2a_3 = {}^2a_2 + {}^1b_2 \quad \text{and} \quad {}^2a_2 = {}^2a_1 + {}^1b_1 = 0;$$

therefore $^2a_3 = 0$, hence $C = 0$, and thus the rest.

Finally, we have $u_n = u_{n-1}p$, therefore $u_n = Cp^n$; now, putting $n = 1$, $u_n = 0$; therefore $C + 0$ and $u_n = 0$; therefore

$$\begin{aligned} {}_n\lambda_x =& (n-1)pq \cdot {}_n\lambda_{x-2} - \frac{(n-2)(n-3)}{1.2}p^2q^2 \cdot {}_n\lambda_{x-4} \\ &+ \frac{(n-3)(n-4)(n-5)}{1.2.3}p^3q^3 \cdot {}_n\lambda_{x-6} - \cdots \\ &+ q \cdot {}_{n+1}\lambda_{x-1} - (n-2)pq^2 \cdot {}_{n+1}\lambda_{x-3} + \frac{(n-3)(n-4)}{1.2}p^2q^3 \cdot {}_{n+1}\lambda_{x-5} \\ &- \cdots \end{aligned}$$

If we make $n = i + m - i$, we will have

$$_{i+m-1}\lambda_x = {}_{m-1}y_x \quad \text{and} \quad {}_{i+m}\lambda_x = 0;$$

therefore

$$(\mathfrak{u}) \quad \begin{cases} {}_{m-1}y_x = (i+m-2)pq \cdot {}_{m-1}y_{x-2} - \dfrac{(i+m-3)(i+m-4)}{1.2}p^2q^2 \cdot {}_{m-1}y_{x-4} \\ \qquad + \dfrac{(i+m-4)(i+m-5)(i+m-6)}{1.2.3}p^2q^2 \cdot {}_{m-1}y_{x-6} - \cdots \end{cases}$$

If therefore we name z_x the probability that A will win before or at trial x, we will have, by a process similar to that of the preceding Problem,

$$(\pi) \qquad z_x = (m+i-2)pqz_{x-2} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2z_{x-4} + \cdots + C.$$

Similarly, if we name $\overset{1}{z}_x$ the probability of player B winning before, or at trial x, we will have

$$(\pi') \qquad \overset{1}{z}_x = (m+i-2)pq\overset{1}{z}_{x-2} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2\overset{1}{z}_{x-4} + \cdots + {}^1C.$$

In order to determine the arbitrary constants which enter into the expressions of z_x and $\overset{1}{z}_x$, I observe that they are to the number of $\frac{m+i}{2}$ if $m+i$ is even, or $\frac{m+i+1}{2}$ if it is odd; now here is in what manner we will have them.

I suppose m and i odd; the equation $(\mathfrak{u})$ will begin visibly to take place only when $x - i - m + 2$ will equal 0, this gives $x = i + m - 2$. The equation (π) will begin to exist therefore only when x will equal $i + m + 1$; it is necessary, consequently, to have all the values of z_x, from z_1 to z_{i+m+1}, in order to determine the arbitrary constants of equation (π).

If m and i are some even numbers, the equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+2$ will equal 1; this gives $x=i+m-1$. The equation (π) begins therefore to take place only when x equals $i+m+2$; it is necessary, consequently, to have the values of z_x, from z_2 to z_{i+m+2}.

If, m being even, i is odd, equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+1$ will equal 1, this gives $x=i+m$. The equation (π) has therefore a place only when x equals $i+m+3$; thus it is necessary to have the values of z_x, from z_2 to z_{i+m+3}.

Finally, if, m being odd, i is even, equation $(\mathfrak{u})$ will begin to take place only when $x-i-m+1$ will equal 0, this gives $x=i+m-1$. Equation (π) begins therefore to exist only when x equals $i+m+2$. It is necessary consequently to have the values of z_x, from z_1 to z_{i+m+2}.

This put, the number of all the possible cases to trial m, each multiplied by their particular probability, will be

$$p^m+mp^{m-1}q+\frac{m(m-1)}{1.2}p^{m-2}q^2+\cdots+q^m.$$

The number of cases which make A win at trial m equals p^m. In order to have the number of cases which make him win precisely at trial $m+2$, it is clear that it is necessary to subtract p^m from the preceding quantity, and to multiply the rest by $p^2+2pq+q^2$, this gives

$$(\chi)\qquad \begin{cases} mp^{m+1}q+\dfrac{m(m-1)}{1.2}p^mq^2+\dfrac{m(m-1)(m-2)}{1.2.3}p^{m-1}q^3+\cdots \\ \qquad +2mp^mq^2+\dfrac{2m(m-1)}{1.2}p^{m-1}q^3+\cdots+mp^{m-1}q^3+\cdots \end{cases}$$

Now, the number of cases which make him win precisely at trial $m+2$ is clearly $mp^{m+1}q$; we have therefore

$$z_{m+2}=p^m(1+mpq).$$

In order to have the number of cases which make A win at trial $m+4$, it is necessary to subtract from the preceding quantity (χ), $mp^{m+1}q$, to multiply the rest by $p^2+2pq+q^2$, and we will have $\frac{m(m+3)}{1.2}p^{m+2}q^2$ for the number of these cases; thus,

$$z_{m+4}=p^m\left[1+mpq+\frac{m(m+3)}{1.2}p^2q^2\right].$$

We will find, likewise,

$$z_{m+6}=p^m\left[1+mpq+\frac{m(m+3)}{1.2}p^2q^2+\frac{m(m+4)(m+5)}{1.2.3}p^3q^3\right],$$

and thus in sequence; the law of these values of z_x holds to z_{m+i-2}; if we have need of further values of z_x, one could obtain them easily by this process.

In order to integrate now the equation (π), it is necessary to have the roots of the equation

$$f^{\frac{m+i-1}{2}} = (m-i-2)pqf^{\frac{m+i-3}{2}} - \frac{(m+i-3)(m+i-4)}{1.2}p^2q^2f^{\frac{m+i-5}{2}} + \cdots,$$

if $m+i$ is odd, or

$$f^{\frac{m+i}{2}-1} = (m-i-2)pqf^{\frac{m+i}{2}-2} - \cdots$$

if $m+i$ is even; now we will find these roots by considering that we have

$$\sin(m+i)z = x\left[2^{m+i-1}u^{m+i-1} - (m+i-2)2^{m+i-3}u^{m+i-3} + \cdots\right],$$

x being the sine and u the cosine of angle z; now, putting

$$\sin(m+i)z = 0,$$

we will have

$$u^{m+i-1} = (m+i-2)\frac{1}{4}u^{m+i-3} - \cdots$$

Let $u = \frac{\sqrt{f}}{2\sqrt{pq}}$, and we will have

$$f^{\frac{m+i-1}{2}} = (m+i-2)pqf^{\frac{m+i-3}{2}} - \cdots$$

if $m+i$ is odd, or

$$f^{\frac{m+i}{2}-1} = (m-i-2)pqf^{\frac{m+i}{2}-2} - \cdots$$

if $m+i$ is even; the different values of u are the cosines of the angles z, such that $\sin(m+i)z$ equals 0, this gives

$$z = \frac{\pi}{m+i}, \quad z = \frac{2\pi}{m+i}, \quad z = \frac{3\pi}{m+i}, \quad \cdots$$

Let l, l_1, $l_2, \ldots$ be the cosines of these angles to $\frac{m+i}{2}$ if $m+i$ is even, or $\frac{m+i-1}{2}$ if it is odd; the different values of f will be $4l^2pq$, $4l_1^2pq, \ldots$. These values one time determined, it is easy to find those of z_x and $\overset{1}{z}_x$.

XXXV.

PROBLEM XIX. — *I suppose two players* A *and* B*, with an equal number m of écus, playing to this condition, that the one who loses will give an écu to the other; let the probability of* A *winning a trial be p; let that of* B *be q; but let it be able to happen that any of them not win, and let the probability of this be r. This put, we ask the probability that the game will end before or at the number x of trials.*

Let ${}_0y_x$ be the number of cases according to which, at the trial x, the gain of the two players is null, multiplied by their probabilities; ${}_1y_x$, ${}_2y_x$, ${}_3y_x, \ldots$ the number of cases according to which the gain of player A is 1, 2, 3, … at trial x, multiplied by their probability, and let ${}_1\overset{1}{y}_x,\ {}_2\overset{1}{y}_x,\ {}_3\overset{1}{y}_x,\ \dots$ express the same things for player B. This put, we will form the following equations:

$$(-)\qquad \begin{cases} {}_0y_x = r\cdot {}_0y_{x-1} + q\cdot {}_1y_{x-1} + p\cdot {}_1\overset{1}{y}_{x-1}, \\ {}_1y_x = r\cdot {}_1y_{x-1} + q\cdot {}_2y_{x-1} + p\cdot {}_0y_{x-1}, \\ {}_2y_x = r\cdot {}_2y_{x-1} + q\cdot {}_3y_{x-1} + p\cdot {}_1y_{x-1}, \\ \vdots \\ {}_ny_x = r\cdot {}_ny_{x-1} + q\cdot {}_{n+1}y_{x-1} + p\cdot {}_{n-1}y_{x-1}, \\ \vdots \\ {}_{m-1}y_x = r\cdot {}_{m-1}y_{x-1} + p\cdot {}_{m-2}y_{x-1} \end{cases}$$

Now we have

$$p\cdot {}_1\overset{1}{y}_{x-1} = q\cdot {}_1y_{x-1};$$

the first equation will become therefore

$${}_0y_x = r\cdot {}_0y_{x-1} + 2q\cdot {}_1y_{x-1};$$

and, if one combines it with the second, we will have

$${}_1y_x = 2r\cdot {}_1y_{x-1} + (2pq - r^2)\,{}_1y_{x-2} + q\cdot {}_2y_{x-1} - qr\cdot {}_2y_{x-2}.$$

Let now

$${}_ny_x = a_n\cdot {}_ny_{x-1} + {}^1a_n\cdot {}_ny_{x-2} + \cdots + u_n + b_n\cdot {}_{n+1}y_{x-1} + {}^1b_n\cdot {}_{n+1}y_{x-3} + \cdots;$$

therefore

$$\begin{aligned} p\cdot {}_{n-1}y_{x-1} = {} & a_{n-1}p\cdot {}_{n-1}y_{x-2} + {}^1a_{n-1}p\cdot {}_{n-1}y_{x-2} + \cdots + pu_{n-1} \\ & + b_{n-1}p\cdot {}_ny_{x-2} + {}^1b_{n-1}p\cdot {}_ny_{x-3} + \cdots \end{aligned}$$

Substituting in place of $p\cdot {}_{n-1}y_{x-1},\ p\cdot {}_{n-1}y_{x-2},\ \dots$ their values that equation $(-)$ gives, we will have

$$\begin{aligned} {}_ny_x = {} & (a_{n-1}+r)\cdot {}_ny_{x-1} + ({}^1a_{n-1} - a_{n-1}r + pb_{n-1})\,{}_ny_{x-2} \\ & + ({}^2a_{n-1} - {}^1a_{n-1}r + p\cdot {}^1b_{n-1})\,{}_ny_{x-3} + \cdots \\ & + q\cdot {}_{n+1}y_{x-1} - a_{n-1}q\cdot {}_{n+1}y_{x-3} - {}^1a_{n-1}q\cdot {}_{n+1}y_{x-5} - \cdots + pu_{n-1}; \end{aligned}$$

whence, by comparing, we will have

$$\begin{aligned} a_n &= a_{n-1} + r, \\ b_n &= q, \\ {}^1a_n &= {}^1a_{n-1} - a_{n-1}r + pb_{n-1}, \\ {}^1b_n &= -a_{n-1}q, \\ {}^2a_n &= {}^2a_{n-1} - {}^1a_{n-1}r + p\cdot {}^1b_{n-1}, \\ &\ \ \vdots \end{aligned}$$

The first of these equations begins to exist when n equals 2; the second, when n equals 1; the third, when n equals 2; etc. We will have therefore, by integrating and adding the appropriate constants,

$$\begin{aligned} a_n &= r(n+1), \\ b_n &= q, \\ {}^1a_n &= -r^2 \cdot \frac{n(n+1)}{1.2} + pq(n+1), \\ {}^1b_n &= -a_{n-1}q = -qrn. \end{aligned}$$

This last equation being true, when n equals 1, it follows that the fifth equation begins to exist when n equals 2; this gives

$${}^2a_n = r^3 \frac{(n+1)n(n-1)}{1.2.3} - pqr(n+1)(n-1).$$

Therefore

$${}^2b_n = qr^2 \frac{n(n-1)}{1.2},$$

an equation which begins to exist when n equals 1, because ${}^2b_1$ equals 0. Therefore, the sixth equation begins to exist when n equals 2, and we will have

$$\begin{aligned} {}^3a_n = &- r^4 \frac{(n+1)n(n-1)(n-2)}{1.2.3.4} \\ &+ pqr^2(n+1)(n-1)(n-2) - p^2q^2 \frac{(n+1)(n-2)}{1.2} + C. \end{aligned}$$

Now, putting $n = 2$, we have

$${}^3a_2 = {}^3a_1 - {}^2a_1 r + p \cdot {}^2b_1 = 0,$$

therefore $C = 0$, and thus in sequence; finally, $u_n = 0$. We will have therefore, by making $n = m-1$ and rejecting the terms ${}_my_{x-1}$, ${}_my_{x-2}$, $\cdots$

$$\begin{aligned} {}_{m-1}y_x = &mr \cdot {}_{m-1}y_{x-1} - \left[r^2 \frac{m(m-1)}{1.2} - pqm \right] {}_{m-1}y_{x-2} \\ &+ \left[r^3 \frac{m(m-1)(m-2)}{1.2.3} - pqrm(m-2) \right] {}_{m-1}y_{x-3} \\ &- \left[r^4 \frac{m(m-1)(m-2)(m-3)}{1.2.3.4} - pqr^2 \frac{m(m-2)(m-3)}{1.2} + p^2q^2 \frac{m(m-3)}{1.2} \right] {}_{m-1}y_{x-4} \\ &+ \cdots \end{aligned}$$

If one supposes $r = 0$, we will have

$${}_{m-1}y_x = mpq \cdot {}_{m-1}y_{x-2} - \frac{m(m-3)}{1.2} p^2q^2 \cdot {}_{m-1}y_{x-4} + \cdots,$$

the same equation as I have found above for that case.

If we name z_x the probability of A winning before or at trial x, we will have

$$z_x = mrz_{x-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] z_{x-2} + \cdots + C,$$

C being an arbitrary constant.

Similarly, if we name $\overset{1}{z}_x$ the probability of B winning before or at trial x, we will have

$$\overset{1}{z}_x = mr\overset{1}{z}_{x-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] \overset{1}{z}_{x-2} + \cdots + \overset{1}{C}.$$

In order to integrate these equations, it is necessary to have the roots of the equation

$$(\Lambda) \qquad f^m = mrf^{m-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] f^{m-2} + \cdots;$$

now here is how one can determine them.

We have seen previously how one could have the roots of the equation

$$y^m = mpqy^{m-2} - \frac{m(m-3)}{1.2}p^2q^2y^{m-4} + \cdots.$$

Let $y = f - r$, and we will have

$$\begin{aligned} f^m = &mrf^{m-1} - \left[r^2\frac{m(m-1)}{1.2} - pqm\right] f^{m-2} \\ &+ \left[r^3\frac{m(m-1)(m-3)}{1.2.3} - pqrm(m-2)\right] f^{m-3} \\ &- \cdots, \end{aligned}$$

an equation which is the same as equation (Λ); the different values of f are consequently equal to those of y, augmented by the quantity r; now the integration of the differential equation in z_x has nothing troublesome.