Théorie Analytique des Probabilités

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Book II §§5–9. pp. 203–228

5. An urn being supposed to contain the number x of balls, we draw from it a part or the totality, and we ask the probability that the number of balls extracted will be even.

The sum of the cases in which this number is unity equals evidently x, since each of the balls can equally be extracted. The sum of the cases in which this number equals 2 is the sum of the combinations of x balls taken two by two, and this sum is, by no. 3, equal to $\frac{x(x-1)}{1.2}$. The sum of the cases in which the same number equals 3 is the sum of the combinations of balls taken three by three, and this sum is $\frac{x(x-1)(x-2)}{1.2.3}$, and thus in sequence. Thus the successive terms of the development of the function $(1+1)^x - 1$ will represent all the cases in which the number of balls extracted is successively $1.2.3\ldots$ to x; whence it is easy to conclude that the sum of all the cases relative to the odd numbers is $\frac{1}{2}(1+1)^x - \frac{1}{2}(1-1)^x,$ or 2^{x-1}, and that the sum of all the cases relative to the even numbers is $\frac{1}{2}(1+1)^x + \frac{1}{2}(1-1)^x,$ or $2^{x-1} - 1$. The reunion of these two sums is the number of all the possible cases; this number is therefore $2^x - 1$; thus the probability that the number of balls extracted will be even is $\frac{2^{x-1}-1}{2^x-1}$, and the probability that this number will be odd is $\frac{2^{x-1}}{2^x-1}$; there is therefore advantage to wager with equality on an odd number.

If the number x is unknown, and if one knows only that it can not exceed n, and that this number and all the lesser are equally possible, we will have the number of all the possible cases relative to the odd numbers by making the sum of all the values of 2^{x-1}, from $x = 1$ to $x = n$, and it is easy to see that this sum is $2^n - 1$. We will likewise have the sum of all the possible cases relative to the even numbers, by summing the function $2^{x-1} - 1$, from $x = 1$ to $x = n$, and we find this sum equal to $2^n - n - 1$; the probability of an even number is therefore then $\frac{2^n-n-1}{2^{n+1}-n-2}$, and that of an odd number is $\frac{2^n-1}{2^{n+1}-n-2}$.

We suppose now that the urn contains the number x of white balls, and the same number of black balls; we ask the probability that by drawing any even number of balls, we will bring forth as many white balls as black balls, all the even numbers being able to be brought forth equally.

The number of cases in which one white ball of the urn can be combined with a black ball is evidently $x.x$. The number of cases in which two white balls can be combined with two black balls is $\frac{x(x-1)}{1.2}\frac{x(x-1)}{1.2}$, and thus in sequence. The number of cases in which we will bring forth as many white balls as black balls is therefore the sum of the squares of the terms of the development of the binomial $(1+1)^x$, less unity. In order to have this sum, we will observe that it is equal to a term independent of a, in the development of $\left(1+\frac{1}{a}\right)^x(1+a)^x$. This function is equal to $\frac{(1+a)^{2x}}{a^x}$. The term independent of a, in its development, is thus the coefficient of the middle term of the binomial $(1+a)^{2x}$; this coefficient is $\frac{1.2.3\dots 2x}{(1.2.3\dots x)^2}$; the number of cases in which we can draw from the urn as many white balls as black balls is therefore

$$\frac{1.2.3\dots 2x}{(1.2.3\dots x)^2} - 1.$$

The number of all possible cases is the sum of the odd terms in the development of the binomial $(1+1)^{2x}$, less the first,or unity. This sum is $\frac{1}{2}(1+1)^{2x} + \frac{1}{2}(1-1)^{2x}$; the number of possible cases is therefore $2^{2x-1} - 1$, which gives for the expression of the probability sought

$$\frac{\frac{1.2.3\dots 2x}{(1.2.3\dots x)^2} - 1}{2^{2x-1} - 1}.$$

In the case where x is a great number, this probability is reduced by no. 33 of Book I to $\frac{2}{\sqrt{x\pi}}$, π being the semi-circumference of which 1 is the radius.

6. We consider a number $x + x'$ of urns, of which the first contains p white balls and q black balls, the second p' white balls and q' black balls, the third p'' white balls and q'' black balls, and thus in sequence. We suppose that we draw successively one ball from each urn. It is clear that the number of all the possible cases in the first drawing is $p + q$; in the second drawing, each of the cases of the first being able to be combined with the $p' + q'$ balls of the second urn, we will

have $(p+q)(p'+q')$ for the number of all the possible cases relative to the first two drawings. In the third drawing, each of these cases can be combined with the $p''+q''$ balls of the third urn; this which gives $(p+q)(p'+q')(p''+q'')$ for the number of all the possible cases relative to the three drawings, and thus of the rest. This product for the totality of the urns will be composed of $x+x'$ factors, and the sum of all the terms of its development in which the letter p, with or without accent, is repeated x times, and consequently the letter q, x' times, will express the number of cases in which we can draw from the urns x white balls and x' black balls.

If p', p'', $\dots$ are equal to p, and if q', q'', $\dots$ are equal to q, the preceding product becomes $(p+q)^{x+x'}$. The term multiplied by $p^x q^{x'}$ in the development of this binomial is

$$\frac{(x+x')(x+x'+1)\cdots(x+1)}{1.2.3\dots x'}p^x q^{x'} \quad \text{or} \quad \frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'}p^x q^{x'}.$$

Thus this quantity expresses the number of cases in which we can bring forth x white balls and x' black balls. The number of all the possible cases being $(p+q)^{x+x'}$, the probability to bring forth x white balls and x' black balls is

$$\frac{1.2.3\dots(x+x')}{1.2.3\dots x.1.2.3\dots x'}\left(\frac{p}{p+q}\right)^x\left(\frac{q}{p+q}\right)^{x'},$$

where we must observe that $\frac{p}{p+q}$ is the probability of drawing a white ball from one of the urns, and that $\frac{q}{p+q}$ is the probability of drawing from it a black ball.

It is clear that it is perfectly equal to draw x white balls and x' black balls from $x+x'$ urns which each contain p white balls and q black balls, or one alone of these urns, provided that we replace into the urn the ball extracted at each drawing.

We consider now a number $x+x'+x''$ urns of which the first contains p white balls, q black balls and r red balls, of which the second contains p' white balls, q' black balls and r' red balls, and thus in sequence. We suppose that we draw one ball from each of these urns. The number of all the possible cases will be the product of the $x+x'+x''$ factors,

$$(p+q+r)(p'+q'+r')(p''+q''+r'')\cdots$$

The number of cases in which we will bring forth x white balls, x' black balls and x'' red balls will be the sum of all the terms of the development of this product, in which the letter p will be repeated x times, the letter q, x' times and the letter r,

x'' times. If all the accented letters p', q', $\ldots$ are equal to their corresponding non-accented, the preceding product is changed into the trinomial $(p+q+r)^{x+x'+x''}$. The term of its development, which has for factor $p^x q^{x'} r^{x''}$, is

$$\frac{1.2.3\ldots(x+x'+x'')}{1.2.3\ldots x.1.2.3\ldots x'.1.2.3\ldots x''} p^x q^{x'} r^{x''};$$

thus, the number of all the possible cases being $(p+q+r)^{x+x'+x''}$, the probability to bring forth x white balls, x' black balls and x'' red balls will be

$$\frac{1.2.3\ldots(x+x'+x'')}{1.2.3\ldots x.1.2.3\ldots x'.1.2.3\ldots x''}\left(\frac{p}{p+q+r}\right)^x\left(\frac{q}{p+q+r}\right)^{x'}\left(\frac{r}{p+q+r}\right)^{x''},$$

whence we must observe that $\frac{p}{p+q+r}$, $\frac{q}{p+q+r}$, $\frac{r}{p+q+r}$ are the respective probabilities of drawing from each urn a white ball, a black ball and a red ball.

We see generally that, if the urns contain each the same number of colors, p being the number of the balls of the first color, q the one of the balls of the second color, r, s, $\ldots$ those of the balls of the third, the fourth, $\ldots$, $x+x'+x''+x'''+\cdots$ being the number of urns, the probability to bring forth x balls of the first color, x' balls of the second, x'' of the third, x''' of the fourth, $\ldots$ will be

$$\frac{1.2.3\ldots(x+x'+x''+x'''+\cdots)}{1.2.3\ldots x.1.2.3\ldots x'.1.2.3\ldots x''.1.2.3\ldots x'''\ldots}\left(\frac{p}{p+q+r+s+\cdots}\right)^x$$
$$\times\left(\frac{q}{p+q+r+s+\cdots}\right)^{x'}\left(\frac{r}{p+q+r+s+\cdots}\right)^{x''}\left(\frac{s}{p+q+r+s+\cdots}\right)^{x'''}\cdots$$

7. We determine now the probability to draw from the preceding urns x white balls, before bringing forth x' black balls, or x'' red balls, $\ldots$. It is clear that, n expressing the number of the colors, this must happen at the latest after $x+x'+x''+\cdots-n+1$ drawings; because, when the number of white balls extracted is equal or less than x, the one of the extracted black balls less than x', the one of the extracted red balls less than x'', $\ldots$, the total number of the extracted balls, and consequently the number of drawings, is equal or less than $x+x'+x''+\cdots-n+1$; we can therefore consider here only $x+x'+x''+\cdots-n+1$ urns.

In order to have the number of cases in which we can bring forth x white balls at the $(x+1)^{\text{st}}$ drawing, it is necessary to determine all the cases in which $x-1$ white balls will be drawn at the drawing $x+i-1$. This number is the term multiplied by p^{x-1} in the development of the polynomial $(p+q+r+\cdots)^{x+i-1}$, and this term is

$$\frac{1.2.3\ldots(x+i-1)}{1.2.3\ldots(x-1)1.2.3\ldots i}p^{x-1}(q+r+\cdots)^i.$$

by combining it with the p white balls of the urn $x+i$, we will have a product which it will be necessary to multiply by the number of all the possible cases relative to the $x'+x''+\cdots-n+1$ following drawings, and this number is

$$(p+q+r+\cdots)^{x'+x''+\cdots-n+1};$$

we will have therefore

$$(a)\qquad \frac{1.2.3\ldots(x+i-1)}{1.2.3\ldots(x-1)1.2.3\ldots i}p^{x-1}(q+r+\cdots)^i(p+q+r+\cdots)^{x'+x''+\cdots-n+1},$$

for the number of cases in which the event can happen precisely at the drawing $x+i$. It is necessary however to exclude the case in which q is raised to the power x', those in which r is raised to the power x'', etc.; because in all these cases it has already happened in the drawing $x+i-1$, either x' black balls, or x'' red balls, or etc. Thus in the development of the polynomial $(q+r+\cdots)^i$, it is necessary to have regard only to the terms multiplied by $q^f r^{f'} s^{f''}\ldots$ in which f is less than x', f' is less than x'', f'' is less than x''', $\ldots$ The term multiplied by $q^f r^{f'} s^{f''}\ldots$ in this development is

$$\frac{1.2.3\ldots i}{1.2.3\ldots f.1.2.3\ldots f'.1.2.3\ldots f''}q^f r^{f'} s^{f''}\ldots$$

All the terms that we must consider in the function (*a*) are therefore represented by

$$(b)\qquad \left\{\begin{aligned} &\frac{1.2.3\ldots(x+f+f'+\cdots-1)}{1.2.3\ldots(x-1).1.2.3\ldots f.1.2.3\ldots f'\ldots}p^x q^f r^{f'}\cdots \\ &\qquad\times(p+q+r+\cdots)^{x'+x''+\cdots-f-f'-\cdots-n+1},\end{aligned}\right.$$

because i is equal to $f+f'+\cdots$. Thus, by giving, in this last function, to f all the whole values from $f=0$ to $f=x'-1$, to f' all the values from $f'=0$ to

$f' = x'' - 1$, and thus in sequence, the sum of all these terms will express the number of cases in which the proposed event can happen in $x + x' + \cdots - n + 1$ drawings. It is necessary to divide this sum by the number of all the possible cases, that is to say by $(p+q+r+\cdots)^{x+x'+x''+\cdots-n+1}$. If we designate by p' the probability of drawing a white ball from any one of the urns, by q' that of drawing from it a black ball, by r' that of drawing a red ball, ..., we will have

$$p' = \frac{p}{p+q+r+\cdots}, \quad q' = \frac{q}{p+q+r+\cdots}, \quad r' = \frac{r}{p+q+r+\cdots}, \quad \cdots;$$

the function (*b*), divided by $(p+q+r+\cdots)^{x+x'+x''+\cdots-n+1}$, will become thus

$$\frac{1.2.3\ldots(x+f+f'+\cdots-1)}{1.2.3\ldots(x-1).1.2.3\ldots f.1.2.3\ldots f'\ldots} p'^x q'^f r'^{f'} \cdots.$$

The sum of the terms which we will obtain by giving to f all the values from $f = 0$ to $f = x' - 1$, to f' all the values from $f' = 0$ to $f' = x' - 1$, will be the sought probability to bring forth x white balls before x' black balls, or x'' red balls, or, etc.

We can, after this analysis, determine the lot of a number n of players A, B, C,..., of whom p', q', r', ... represent the respective skills, that is to say their probabilities to win a trial when, in order to win the game, there lacks x trials to player A, x' trials to player B, x'' trials to player C, and thus in sequence; because it is clear that, relatively to player A, this reverts to determine the probability to bring forth x white balls before x' black balls, or x'' red balls, ..., by drawing successively a ball from a number $x + x' + x'' + \cdots - n + 1$ of urns which contain each p white balls, q black balls, r red balls, $\cdots$, p, q, r, ... being respectively equal to the numerators of the fractions p', q', r', ... reduced to the same denominator.

8. The preceding problem can be resolved in a quite simple manner by the analysis of the generating functions. We name $y_{x,x',x'',\ldots}$ the probability of player A to win the game. At the following trial, this probability is changed into $y_{x-1,x',x'',\ldots}$, if A wins this trial, and the probability for this is p'. The same probability is changed into $y_{x,x'-1,x'',\ldots}$, if the trial is won by player B, and the probability for this is q'; it is changed into $y_{x,x',x''-1,\ldots}$ if the trial is won by player C, and the probability for this is r', and thus in sequence; we have therefore the equation in the partial differences

$$y_{x,x',x'',\ldots} = p'y_{x-1,x',x'',\ldots} + q'y_{x,x'-1,x'',\ldots} + r'y_{x,x',x''-1,\ldots} + \cdots$$

Let u be a function of t, t', t'', $\ldots$, such that $y_{x,x',x'',\ldots}$ is the coefficient of $t^x t'^{x'} t''^{x''} \ldots$ in its development; the preceding equation in the partial differences will give, by passing from the coefficients to the generating functions,

$$u = u(p't + q't' + r't'' + \cdots),$$

whence we deduce

$$1 = p't + q't' + r't'' + \cdots;$$

consequently,

$$\frac{1}{t} = \frac{p'}{1 - q't' - r't'' - \cdots},$$

this which gives

$$\frac{u}{t^x} = \frac{up'^x}{(1 - q't' - r't'' - \cdots)^x} = up'^x \left\{ \begin{array}{l} 1 + x(q't' + r't'' + \cdots) \\ \quad + \dfrac{x(x+1)}{1.2}(q't' + r't'' + \cdots)^2 \\ \quad + \dfrac{x(x+1)(x+2)}{1.2.3}(q't' + r't'' + \cdots)^3 \\ \quad + \cdots \end{array} \right\}.$$

Now the coefficient of $t^0 t'^{x'} t''^{x''} \ldots$ in $\frac{u}{t^x}$ is $y_{x,x',x'',\ldots}$, and the same coefficient in any term of the last member of the preceding equation, such as $kup'^x t'^{l'} t''^{l''} \ldots$, is $kp'^x y_{0,x'-t',x''-t'',\ldots}$; the quantity $y_{0,x'-t',x''-t'',\ldots}$ is equal to unity, since then player A lacks no coup. Moreover, it is necessary to reject all the values of $y_{0,x'-t',x''-t'',\ldots}$ in which l' is equal or greater than x', l'' is equal or greater than x'', and thus in sequence, because these terms are not able to be given by the equation in the partial differences, the game being finite, when any one of the players B, C, $\ldots$ have no more coups to play; it is necessary therefore to consider in the last member of the preceding equation only the powers of t' less than x', only the powers of t'' less than x'', $\ldots$. The preceding expression of $\frac{u}{t^x}$ will give thus, by passing again from the generating functions to the coefficients,

$$y_{x,x'-t',x''-t'',\dots} = p'^x \left\{ \begin{array}{l} 1 + x(q't' + r't'' + \cdots) \\ \quad + \dfrac{x(x+1)}{1.2}(q't' + r't'' + \cdots)^2 \\ \quad + \dfrac{x(x+1)(x+2)}{1.2.3}(q't' + r't'' + \cdots)^3 \\ \quad + \cdots \end{array} \right\},$$

provided that we reject the terms in which the power of q' surpasses $x'-1$, those in which the power of r' surpasses $x''-1$, etc. The second member of this equation is developed in one sequence of terms contained in the general formula

$$\frac{1.2.3\dots(x+f+f'+\cdots-1)}{1.2.3\dots(x-1).1.2.3\dots f.1.2.3\dots f'\dots} p'^x q'^f r'^{f'} \dots$$

The sum of these terms relative to all the values of f from f null to $f = x'-1$, to all the values of f' from f' null to $f' = x''-1$, $\dots$, will be the probability $y_{x,x'-t',x''-t'',\dots}$, this which is conformed to that which precedes.

In the case of two players A and B, we will have, for the probability of player A,

$$p'^x \left[1 + xq'' + \frac{x(x+1)}{1.2} q'^2 + \cdots + \frac{x(x+1)(x+2)\cdots(x+x'-2)}{1.2.3\dots(x'-1)} q'^{x'-1} \right].$$

By changing p' into q' and x into x', and reciprocally, we will have

$$q'^{x'} \left[1 + x'p' + \frac{x(x+1)}{1.2} p'^2 + \cdots + \frac{x'(x'+1)(x'+2)\cdots(x+x'-2)}{1.2.3\dots(x'-1)} p'^{x-1} \right]$$

for the probability that player B will win the game. The sum of these two expressions must be equal to unity, this which we see evidently by giving them the following forms. The first expression can, by No. 37 of Book I, is transformed into this one

$$p'^{x+x'-1} \left\{ \begin{array}{l} 1 + \dfrac{x+x'-1}{1} \dfrac{q'}{p'} + \dfrac{(x+x'-1)(x+x'-2)}{1.2} \dfrac{q'^2}{p'^2} + \cdots \\ \quad + \dfrac{(x+x'-1)\cdots(x+1)}{1.2.3\cdots(x'-1)} \dfrac{q'^{x'-1}}{p'^{x'-1}} \end{array} \right\},$$

and the second can be transformed into this one

$$q'^{x+x'-1}\left\{\begin{array}{l} 1+\dfrac{x+x'-1}{1}\dfrac{p'}{q'}+\dfrac{(x+x'-1)(x+x'-2)}{1.2}\dfrac{p'^2}{q'^2}+\cdots \\ \quad+\dfrac{(x+x'-1)\cdots(x+1)}{1.2.3\cdots(x'-1)}\,\dfrac{p'^{x'-1}}{q'^{x'-1}} \end{array}\right\}.$$

The sum of these expressions is the development of the binomial $(p'+q')^{x+x'-1}$, and consequently it is equal to unity, because, A or B must win each trial, the sum $p'+q'$ of their probabilities for this is unity.

The problem which we just resolved is the one which we name the problem of points in the Analysis of chances. The chevalier de Méré proposed it to Pascal, with some other problems on the game of dice. Two players of whom the skills are equal have put into the game the same sum; they must play until one of them has beat a given number of times his adversary; but they agree to quit the game, when there lacks yet x points to the first player in order to attain this given number, and when there lacks x' points to the second player. We demand in what way they must share the sum put into the game. Such is the problem that Pascal resolved by means of his arithmetic triangle. He proposed it to Fermat who gave the solution to it by way of combinations, this which caused between these two great geometers a discussion, to the continuation of which Pascal recognized the goodness of the method of Fermat, for any number of players. Unhappily we have only one part of their correspondence, in which we see the first elements of the theory of probabilities and their application to one of the most curious problems of this theory.

The problem proposed by Pascal to Fermat reverts to determine the respective probabilities of the players in order to win the game; because it is clear that the stake must be shared between the players proportionally to their probabilities. These probabilities are the same as those of two players A and B, who must attain a given number of points, x being the number of those which player A lacks, and x' being the number of those which player B lacks, by imagining an urn containing two balls of which one is white and the other black, both carrying the no. 1, the white ball being for player A, and the black ball for player B. We draw successively one of these balls, and we return it into the urn after each drawing. By naming $y_{x,x'}$ the probability that player A will attain, the first, the given number of points, or, that which reverts to the same, that he will have x points before B has x', we will have

$$y_{x,x'} = \frac{1}{2}y_{x-1,x'} + \frac{1}{2}y_{x,x'-1};$$

because, if the ball that we extract is white, $y_{x,x'}$ is changed into $y_{x-1,x'}$, and if the ball extracted is black, $y_{x,x'}$ is changed into $y_{x,x'-1}$, and the probability of each of these events is $\frac{1}{2}$; we have therefore the preceding equation.

The generating function of $y_{x,x'}$ in this equation in the partial differences is, by No. 20 of Book I,

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'},$$

M being an arbitrary function of t'. In order to determine it, we will observe that $y_{0,0}$ can not have place, since the game ceases when one or the other of the variables x and x' is null; M must therefore have for factor t'. Moreover $y_{0,x'}$ is unity, whatever be x', the probability of player A is changing then into certitude; now the generating function of unity is generally $\frac{t'^i}{1-t'}$, because the coefficients of the powers of t' in the development of this function are all equal to unity; in the present case, $y_{0,x'}$ being able to have place when x' is either 1, or 2, or 3, etc., i must be equal to unity; the generating function of $y_{0,x'}$ is therefore equal to $\frac{t'}{1-t'}$; this is the coefficient of t^0 in the development of the generating function of $y_{x,x'}$ or in

$$\frac{M}{1 - \frac{1}{2}t - \frac{1}{2}t'};$$

we have therefore

$${\frac{M}{1 - \frac{1}{2}t}}' = \frac{t'}{1-t'},$$

this which gives

$$M = \frac{t'(1 - \frac{1}{2}t')}{(1-t')},$$

consequently the generating function of $y_{x,x'}$ is

$$\frac{t'(1-\frac{1}{2}t')}{(1-t')(1-\frac{1}{2}t-\frac{1}{2}t')}.$$

By developing it with respect to the powers of t, we have

$$\frac{t'}{1-t'}\left(1+\frac{1}{2}\,\frac{t}{1-\frac{1}{2}t'}+\frac{1}{2^2}\,\frac{t^2}{(1-\frac{1}{2}t')^2}+\frac{1}{2^3}\,\frac{t^3}{(1-\frac{1}{2}t')^3}+\cdots\right).$$

The coefficient of t^x in this series is

$$\frac{1}{2^x}\,\frac{t'}{(1-t')(1-\frac{1}{2}t')^x};$$

$y_{x,x'}$ is therefore the coefficient of $t'^{x'}$ in this last quantity; now we have

$$\begin{aligned}&\frac{t'}{(1-t')(1-\frac{1}{2}t')^x}\\ &\quad=\frac{t'+\frac{1}{2}x\,t'^2+\frac{1}{2^2}\,\frac{x(x+1)}{2}\,t'^3+\cdots+\frac{1}{2^{x'-1}}\,\frac{x(x+1)(x+2)\cdots(x+x'-2)}{1.2.3\ldots(x'-1)}\,t'^{x'}+\cdots}{1-t'}.\end{aligned}$$

By reducing into series the denominator of this last fraction and multiplying the numerator by this series, we see that the coefficient of $t'^{x'}$ in this product is that which this numerator becomes when we make $t'=1$; we have therefore

$$y_{x,x'}=\left\{\begin{aligned}&1+x\frac{1}{2}+\frac{x(x+1)}{1.2}\,\frac{1}{2^2}+\frac{x(x+1)(x+2)}{1.2.3}\,\frac{1}{2^3}+\cdots\\ &\quad+\frac{x(x+1)\cdots(x+x'-2)}{1.2.3\ldots(x'-1)}\,\frac{1}{2^{x'-1}}\end{aligned}\right\},$$

a result conformed to that which precedes.

We imagine presently that there is in the urn a white ball carrying the no. 1, and two black balls, of which one carries the no. 1, and the other carries the no. 2, the white ball being favorable to A, and the black balls to his adversary, each ball diminishing by its value the number of points which lack to the player to which it is favorable. $y_{x,x'}$ being always the probability that player A will attain first the given number, we will have the equation in the partial differences

$$y_{x,x'} = \frac{1}{3}\,y_{x-1,x'} + \frac{1}{3}\,y_{x,x'-1} + \frac{1}{3}\,y_{x,x'-2};$$

because, in the following drawing, if the white balls exits, $y_{x,x'}$ becomes $y_{x-1,x'}$; if the black ball numbered 1 exits, $y_{x,x'}$ becomes $y_{x,x'-1}$, and if the black ball numbered 2 exits, $y_{x,x'}$ becomes $y_{x,x'-2}$, and the probability of each of these events is $\frac{1}{3}$.

The generating function of $y_{x,x'}$ is

$$\frac{M}{1 - \frac{1}{3}t - \frac{1}{3}t' - \frac{1}{3}t'^2},$$

M being an arbitrary function of t', and in the present case is equal to

$$\frac{t'}{1-t'}(1 - \frac{1}{3}t' - \frac{1}{3}t'^2),$$

so that the generating function of $y_{x,x'}$ is

$$\frac{t'(1 - \frac{1}{3}t' - \frac{1}{3}t'^2)}{(1-t')(1 - \frac{1}{3}t - \frac{1}{3}t' - \frac{1}{3}t'^2)}.$$

The coefficient of t^x in the development of this function is

$$\frac{1}{3^x}\,\frac{t'}{1-t'}\,\frac{1}{(1 - \frac{1}{3}t - \frac{1}{3}t' - \frac{1}{3}t'^2)^x},$$

and there results from this that we come to say that the coefficient of $t'^{x'}$ in the development of this last quantity is equal to

$$\frac{1}{3^x}\left\{\begin{aligned} t' + \frac{xt'^2(1+t')}{3} + \frac{x(x+1)}{1.2}\,\frac{t'^3(1+t')^2}{3^2} \\ + \frac{x(x+1)(x+2)}{1.2.3}\,\frac{t'^4(1+t')^3}{3^3} + \cdots \end{aligned}\right\};$$

by rejecting from the development in this series all the powers of t' superior to $t'^{x'}$, and supposing in this that we conserve $t' = 1$, this will be the expression of $y_{x,x'}$.

It is easy to translate this process into formulae. Thus, by supposing x' even and equal to $2r + 2$, we find

$$
\begin{aligned}
y_{x,x'} = \frac{1}{3^x}&\left[1 + x\,\frac{2}{3} + \frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2 + \cdots + \frac{x(x+1)\cdots(x+r-1)}{1.2.3\ldots r}\left(\frac{2}{3}\right)^r\right] \\
&+ \frac{x(x+1)\cdots(x+r)}{1.2.3\ldots(r+1)\,3^{x+r+1}}\left[1 + (r+1) + \frac{(r+1)r}{1.2} + \cdots + \frac{(r+1)r\ldots 2}{1.2.3\ldots r}\right] \\
&+ \frac{x(x+1)\cdots(x+r+1)}{1.2.3\ldots(r+2)\,3^{x+r+2}}\left[1 + (r+2) + \cdots + \frac{(r+2)(r+1)\ldots 4}{1.2.3\ldots(r-1)}\right] \\
&+ \cdots \\
&+ \frac{x(x+1)\cdots(x+2r)}{1.2.3\ldots(2r+1)\,3^{x+2r+1}}.
\end{aligned}
$$

If we suppose x' odd and equal to $2r+1$, we will have

$$
\begin{aligned}
y_{x,x'} = \frac{1}{3^x}&\left[1 + x\,\frac{2}{3} + \frac{x(x+1)}{1.2}\left(\frac{2}{3}\right)^2 + \cdots + \frac{x(x+1)\cdots(x+r-1)}{1.2.3\ldots r}\left(\frac{2}{3}\right)^r\right] \\
&+ \frac{x(x+1)\cdots(x+r)}{1.2.3\ldots(r+1)\,3^{x+r+1}}\left[1 + (r+1) + \frac{(r+1)r}{1.2} + \cdots + \frac{(r+1)r\ldots 3}{1.2.3\ldots(r-1)}\right] \\
&+ \frac{x(x+1)\cdots(x+r+1)}{1.2.3\ldots(r+2)\,3^{x+r+2}}\left[1 + (r+2) + \frac{(r+2)(r+1)}{1.2} + \cdots + \frac{(r+2)(r+1)\ldots 5}{1.2.3\ldots(r-2)}\right] \\
&+ \cdots \\
&+ \frac{x(x+1)\cdots(x+2r-1)}{1.2.3\ldots(2r+1)\,3^{x+2r}}.
\end{aligned}
$$

Thus, in the case of $x = 2$ and $x' = 5$, we have

$$y_{2,5} = \frac{350}{729}.$$

We imagine further that there are in the urn two distinguished white balls, as the two black balls, by the nos. 1 and 2; the probability of player A will be given by the equation in the partial differences

$$y_{x,x'} = \frac{1}{4}y_{x-1,x'} + \frac{1}{4}y_{x-2,x'} + \frac{1}{4}y_{x,x'-1} + \frac{1}{4}y_{x-1,x'-2}.$$

The generating function of $y_{x,x'}$ is then, by No. 20 of Book I,

$$\frac{M+Nt}{1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2},$$

M and N being two arbitrary functions of t'. In order to determine them, we will observe that $y_{0,x'}$ is always equal to unity, and that it is necessary to exclude in M the null power of t'; we have therefore

$$M=\frac{t'}{1-t'}(1-\frac{1}{4}t'-\frac{1}{4}t'^2).$$

In order to determine N, we seek the generating function of $y_{1,x'}$. If we observe that $y_{0,x'}$ is equal to unity, and that, player A having no more need of a point, he wins the game, either that he brings forth the white ball numbered 1 or the white ball numbered 2, the preceding equation in the partial differences will give

$$y_{1,x'}=\frac{1}{2}+\frac{1}{4}y_{1,x'-1}+\frac{1}{4}y_{1,x'-2}.$$

We suppose $y_{1,x'}=1-y'_{x'};$ we will have

$$y'_{x'}=\frac{1}{4}y'_{x'-1}+\frac{1}{4}y'_{x'-2}.$$

The generating function of this equation is

$$\frac{m+nt'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2},$$

m and n being two constants. In order to determine them, we will observe that $y_{1,0}=0$, and that consequently $y'_0=1$, this which gives $m=1$. The generating function of $y'_{x'}$ is therefore

$$\frac{1+nt'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2}.$$

We have next evidently $y_{1,1}=\frac{1}{2}$, this which gives $y'_1=\frac{1}{2};$ y'_1 is the coefficient of t' in the development of the preceding function, and this coefficient is $n+\frac{1}{4}$; we have therefore $n+\frac{1}{4}=\frac{1}{2}$, or $n=\frac{1}{4}$. The generating function of unity is $\frac{1}{1-t'}$, because here all the powers of t' can be admitted; we have thus

$$\frac{1}{1-t'} - \frac{1+\frac{1}{4}t'}{1-\frac{1}{4}t'-\frac{1}{4}t'^2}, \quad \text{or} \quad \frac{\frac{1}{2}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)},$$

for the generating function of $y_{1,x'}$. This same function is the coefficient of t in the development of the generating function of $y_{x,x'}$, a function which, by that which precedes, is

$$\frac{\frac{t'}{1-t'}(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+Nt}{1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2};$$

this coefficient is

$$\frac{\frac{1}{4}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)} + \frac{N}{1-\frac{1}{4}t'-\frac{1}{4}t'^2};$$

by equating it to

$$\frac{\frac{1}{2}t'}{(1-t')(1-\frac{1}{4}t'-\frac{1}{4}t'^2)},$$

we will have

$$N = \frac{\frac{1}{4}t'}{1-t'}.$$

The generating function of $y_{x,x'}$ is thus

$$\frac{t'(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+\frac{1}{4}tt'}{(1-t')(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)}.$$

If we develop into series the function

$$\frac{t'(1-\frac{1}{4}t'-\frac{1}{4}t'^2)+\frac{1}{4}tt'}{(1-t')(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)} - t',$$

we will have

$$\frac{(2+t)tt'}{4}\left\{\begin{array}{l} 1+\frac{1}{4}t'(1+t')+\frac{1}{4^2}t'^2(1+t')^2+\frac{1}{4}^3 t'^3(1+t')^3+\cdots \\ \quad +\frac{t(1+t)}{4}\left[1+\frac{2}{4}t'(1+t')+\frac{3}{4^2}t'^2(1+t')^2+\frac{4}{4^8}t'^3(1+t')^3+\cdots\right] \\ \quad +\frac{t^2(1+t)^2}{4^2}\left[1+\frac{3}{4}t'(1+t')+\frac{3.4}{1.2.4^2}t'^2(1+t')^2+\frac{3.4.5}{1.2.3.4^3}t'^3(1+t')^3+\cdots\right] \\ \quad +\frac{t^3(1+t)^3}{4^3}\left[1+\frac{4}{4}t'(1+t')+\frac{4.5}{1.2.4^2}t'^2(1+t')^2+\frac{4.5.6}{1.2.3.4^3}t'^3(1+t')^3+\cdots\right] \\ \quad +\cdots \end{array}\right\}.$$

If we reject in this series all the powers of t other than t^x and all the powers of t' superior to $t'^{x'}$, and if in that which remains we make $t=1$, $t'=1$, we will have the expression of $y_{x,x'}$ when x is equal or greater than unity; when x is null, we have $y_{0,x'}=1$. It is easy to translate this process into formulae, as we have made for the preceding case.

We name $z_{x,x'}$ the probability of player B; the generating function of $z_{x,x'}$ will be that which the generating function of $y_{x,x'}$ becomes when we change in it t into t', and reciprocally, this which gives, for this function,

$$\frac{t(1-\frac{1}{4}t-\frac{1}{4}t^2)+\frac{1}{4}tt'}{(1-t)(1-\frac{1}{4}t-\frac{1}{4}t'-\frac{1}{4}t^2-\frac{1}{4}t'^2)}.$$

By adding the two generating functions, their sum is reduced to

$$\frac{t}{1-t}+\frac{t'}{1-t'}+\frac{tt'}{(1-t)(1-t')},$$

in which the coefficient of $t^x t'^{x'}$ is unity; thus we have

$$y_{x,x'}+z_{x,x'}=1,$$

this which is clear besides, since the game must be necessarily won by one of the players.

9. We imagine in an urn r balls marked with the n° 1, r balls marked with n° 2, r balls marked with n° 3, and so on in sequence to the n° n. These balls being well mixed in the urn, one draws them successively; one requires the probability

that there will come forth at least one of these balls at the rank[1] indicated by its label[2], or that there will come forth of them at least two, or at least three, etc.

We seek first the probability that there will come forth at least one of them. For this, we will observe that each ball can come forth at its rank only in the first n drawings; one can therefore here set aside the following drawings; now the total number of balls being rn, the number of their combinations n by n, by having regard for the order that they observe among themselves, is, by that which precedes,

$$rn(rn-1)(rn-2)\cdots(rn-n+1);$$

this is therefore the number of all possible cases in the first n drawings.

We consider one of the balls marked with the n° 1, and we suppose that it comes forth at its rank, or the first. The number of combinations of the $rn-1$ other balls taken $n-1$ by $n-1$ will be

$$(rn-1)(rn-2)\cdots(rn-n+1);$$

this is the number of cases relative to the assumption that we just made, and, as this assumption can be applied to r balls marked with n° 1, one will have

$$r(rn-1)(rn-2)\cdots(rn-n+1)$$

for the number of cases relative to the hypothesis that one of the balls marked with the n° 1 will come forth at its rank. The same result takes place for the hypothesis that any one of the $n-1$ other kinds of balls will come forth at the rank indicated by its label. By adding therefore all the results relative to these diverse hypotheses, one will have

$$(a) \qquad rn(rn-1)(rn-2)\cdots(rn-n+1);$$

for the number of cases in which one ball at least will come forth at its rank, provided however that one removes from them the cases which are repeated.

In order to determine these cases, we consider one of the balls of the n° 1, coming forth first, and one of the balls of the n° 2, coming forth second. This

[1] *Translator's note*: This means that a ball marked with 1 will be drawn first, a ball marked with 2 will be drawn second, and so on. In other words, balls will be drawn consecutively by number.

[2] *Translator's note*: The word here is *numéro*, number. However, this refers to the use of a number as a label. In order to distinguish it from *nombre*, number or quantity, I choose to render it as such.

case is contained twice in the preceding number; for it is contained one time in the number of the cases relative to the assumption that one of the balls labeled[3] 1 will come forth at its rank, and a second time in the number of cases relative to the assumption that one of the balls labeled 2 will come forth at its rank; and, as this extends to any two balls coming forth at their rank, one sees that it is necessary to subtract from the number of the cases preceding the number of all the cases in which two balls come forth at their rank.

The number of combinations of two balls of different labels is $\frac{n(n-1)}{1.2}r^2$; for the number of the labels being n, their combinations two by two are in number $\frac{n(n-1)}{1.2}$, and in each of these combinations one can combine the r balls marked with one of the labels with the r balls marked with the other label. The number of combinations of the $rn-2$ balls remaining, taken $n-2$ by $n-2$, by having regard for the order that they observe among themselves, is

$$(rn-2)(rn-3)\cdots(rn-n+1);$$

thus the number of cases relative to the assumption that two balls come forth at their rank is

$$\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\cdots(rn-n+1);$$

subtracting from it the number (*a*), one will have

$$(a') \qquad \left\{ \begin{aligned} & rn(rn-1)(rn-2)\cdots(rn-n+1) \\ & \quad -\frac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\cdots(rn-n+1), \end{aligned} \right.$$

for the number of all the cases in which one ball at least will come forth at its rank, provided that one subtracts again from this function the repeated cases, and that one adds to them those which are lacking.

These cases are those in which three balls come forth at their rank. By naming k this number, it is repeated three times in the first term of the function (*a'*); for it can result, in this term, from three assumptions of each of the three balls coming forth at its rank. The number k is likewise contained three times in the second term of the function; for it can result from each of the assumptions relative to any two of the three balls coming forth at their rank. Thus, this second

[3] *Translator's note*: The word is *numérotées*, numbered. I have chosen to render it as such for the same reason as above.

term being affected with the $-$ sign, the number k in not found in the function (a'); it is necessary therefore to add it to (a') in order that it contain all the cases in which one ball at least comes forth at its rank. The number of combinations of n labels taken three by three is $\frac{n(n-1)(n-2)}{1.2.3}$, and, as one can combine the r balls of one of these labels of each combination with the r balls of the second label and with the r balls of the third label, one will have the total number of combinations in which three balls come forth at their rank, by multiplying $\frac{n(n-1)(n-2)}{1.2.3}r^3$ by $(rn-3)(rn-4)\cdots(rn-n+1)$, a number which expresses that of the combinations of the $rn-3$ balls remaining, taken $n-3$ by $n-3$, by having regard for the order that they observe among themselves. If one adds this product to the function (a'), one will have

$$(a'') \qquad \left\{\begin{array}{l} rn(rn-1)(rn-2)\cdots(rn-n+1) \\ \quad -\dfrac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\cdots(rn-n+1), \\ \quad +\dfrac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\cdots(rn-n+1). \end{array}\right.$$

This function expresses the number of all cases in which one ball at least comes forth at its rank, provided that one subtracts from it again the repeated cases. These cases are those in which four balls come forth at their rank. By applying here the preceding reasonings, one will see that it is necessary again to subtract from the function (a'') the term

$$\frac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\cdots(rn-n+1).$$

By continuing thus, one will have, for the expression of the cases in which one ball at least comes forth at its rank,

$$(\mathrm{A}) \quad \left\{\begin{array}{l} rn(rn-1)(rn-2)\cdots(rn-n+1) \\ \quad -\dfrac{n(n-1)}{1.2}r^2(rn-2)(rn-3)\cdots(rn-n+1) \\ \quad +\dfrac{n(n-1)(n-2)}{1.2.3}r^3(rn-3)(rn-4)\cdots(rn-n+1) \\ \quad -\dfrac{n(n-1)(n-2)(n-3)}{1.2.3.4}r^4(rn-4)(rn-5)\cdots(rn-n+1) \\ \quad +\cdots \end{array}\right.$$

a series being continued as far at it can be. In this function, each combination is not repeated: thus the combination of s balls coming forth at their rank is found here only one time; for this combination is contained s times in the first term of the function, since it can result from each of the s balls coming forth at its rank; it is subtracted $\frac{s(s-1)}{1.2}$ times in the second term, since it can result from two by two combinations of the s balls coming forth at their rank; it is added $\frac{s(s-1)(s-2)}{1.2.3}$ times in the third term, since it can result from the combinations of s letters taken three by three, and so in sequence; it is therefore, in the function (A), contained a number of times equal to

$$s - \frac{s(s-1)}{1.2} + \frac{s(s-1)(s-2)}{1.2.3} - \cdots,$$

and consequently equal to $1-(1-1)^s$, or to unity. By dividing the function (A) by the number $rn(rn-1)(rn-2)\cdots(rn-n+1)$ of all possible cases, one will have, for the expression of the probability that one ball at least will come forth at its rank,

$$\text{(B)} \qquad \begin{cases} 1 - \dfrac{(n-1)r}{1.2(rn-1)} + \dfrac{(n-1)(n-2)r^2}{1.2.3(rn-1)(rn-2)} \\ \quad - \dfrac{(n-1)(n-2)(n-3)r^3}{1.2.3.4(rn-1)(rn-2)(rn-3)} + \cdots \end{cases}$$

We seek now the probability that s balls at least will come forth at their rank. The number of cases in which s balls come forth a their rank is, by that which precedes,

$$(b) \quad \frac{n(n-1)(n-2)\cdots(n-s+1)}{1.2.3\dots s} r^s (rn-s)(rn-s-1)\cdots(rn-n+1),$$

provided that one subtracts from this function the cases which are repeated. These cases are those in which $s+1$ balls come forth at their rank, for they can result, in the function, from $s+1$ balls taken s by s; these cases are therefore repeated $s+1$ times in this function; consequently it is necessary to subtract them s times. Now the number of cases in which $s+1$ balls come forth at their rank is

$$\frac{n(n-1)(n-2)\cdots(n-s)}{1.2.3\dots(s+1)} r^{s+1} (rn-s-1)(rn-s-2)\cdots(rn-n+1).$$

By multiplying it by s and subtracting it from the function (b), one will have

$$(b')\left\{\begin{array}{l}\dfrac{n(n-1)(n-2)\cdots(n-s+1)}{1.2.3\ldots s}r^s(rn-s)(rn-s-1)\cdots(rn-n+1)\\ \qquad\times\left[1-\dfrac{s(n-s)r}{(s+1)(rn-s)}\right].\end{array}\right.$$

In this function, many cases are again repeated, namely, those in which $s+2$ balls come forth at their rank; for they result, in the first term, from $s+2$ balls coming forth at their rank and taken s by s; they result, in the second term, from $s+2$ balls coming forth at their rank and taken $s+1$ by $s+1$, and moreover multiplied by the factor s, by which one has multiplied the second term. They are therefore contained in this function the number of times $\frac{(s+2)(s+1)}{1.2}-s(s+2)$; thus it is necessary to multiply by unity, less this number of times, the number of cases in which $s+2$ balls come forth at their rank. This last number is

$$\frac{n(n-1)(n-2)\cdots(n-s-1)}{1.2.3\ldots(s+2)}r^{s+2}(rn-s-2)(rn-s-3)\cdots(rn-n+1);$$

the product in question will be therefore

$$\frac{n(n-1)(n-2)\cdots(n-s-1)}{1.2.3\ldots(s+2)}r^{s+2}(rn-s-2)\cdots(rn-n+1)\frac{s(s+1)}{1.2}.$$

By adding it to the function (b'), one will have

$$(b'')\left\{\begin{array}{l}\dfrac{n(n-1)(n-2)\cdots(n-s+1)}{1.2.3\ldots s}r^s(rn-s)(rn-s-1)\cdots(rn-n+1)\\ \qquad\times\left\{\begin{array}{l}1-\dfrac{s}{(s+1)}\,\dfrac{(n-s)r}{(rn-s)}\\ \quad+\dfrac{s}{s+2}\,\dfrac{(n-s)(n-s-1)r^2}{1.2(rn-s)(rn-s-1)}\end{array}\right\}.\end{array}\right.$$

This is the number of all possible cases in which s balls come forth at their rank, provided that one subtracts from it again the cases which are repeated. By continuing to reason so, and by dividing the final function by the number of all possible cases, one will have, for the expression of the probability that s balls at least will come forth at their rank,

$$(C)\quad \left\{ \begin{aligned} &\frac{(n-1)(n-2)\cdots(n-s+1)r^{s-1}}{1.2.3\ldots s(rn-1)(rn-2)\cdots(rn-s+1)} \\ &\qquad \times \left\{ \begin{aligned} &1 - \frac{s}{s+1}\,\frac{(n-s)r}{rn-s} + \frac{s}{s+2}\,\frac{(n-s)(n-s-1)r^2}{1.2.(rn-s)(rn-s-1)} \\ &\quad - \frac{s}{s+3}\,\frac{(n-s)(n-s-1)(n-s-2)r^3}{1.2.3(rn-s)(rn-s-1)(rn-s-2)} + \cdots \end{aligned} \right\}. \end{aligned} \right.$$

One will have the probability that none of the balls will come forth at its rank by subtracting the formula (B) from unity, and one will find, for its expression,

$$\frac{(1.2.3\ldots rn) - nr[1.2.3\ldots(rn-1)] + \frac{n(n-1)}{1.2}r^2[1.2.3\ldots(rn-2)] - \cdots}{1.2.3\ldots rn}.$$

One has, by n° 33[4] of Book I, whatever be i,

$$1.2.3\ldots i = \int x^i dx c^{-x},$$

the integral[5] being taken from x null to x infinity. The preceding expression can therefore be put under this form

$$(o)\qquad \frac{\int x^{rn-n}dx(x-r)^n c^{-x}}{\int x^{rn}dx c^{-x}}.$$

We suppose the number rn of balls in the urn very great; then, by applying to the preceding integrals the method of n° 24[6] of Book I, one will find very nearly for the integral of the numerator,

$$\frac{\sqrt{2\pi}X^{rn+2}\left(1-\frac{r}{X}\right)^{n+1}c^{-X}}{\sqrt{nX^2+n(r-1)(X-r)^2}},$$

X being the value of x which renders a maximum the function $x^{rn-n}(x-r)^n c^{-x}$. The equation relative to this maximum gives for X the two values

[4] pages 128–137.
[5] *Translator's note*: The constant c denotes e, the base of the natural logarithm.
[6] pages 94–96.

$$X = \frac{rn + r}{2} \pm \frac{\sqrt{r^2(n-1)^2 + 4rn}}{2}.$$

One can consider here only the greatest of these values which is, to the quantities nearly of the order $\frac{1}{rn}$, equal to $rn + \frac{n}{n-1}$; then the integral of the numerator of the function (*o*) becomes nearly

$$\frac{\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn}\left(1 - \frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2 + 1}}.$$

The integral of the denominator of the same function is, by n° 33, quite nearly,

$$\sqrt{2\pi}(rn)^{rn+\frac{1}{2}}c^{-rn};$$

the function (*o*) becomes thus

$$\frac{\left(1 - \frac{1}{n}\right)^{n+1}\sqrt{r}}{\sqrt{(r-1)(1-\frac{1}{n})^2 + 1}}.$$

One can put it under the form

$$\frac{\left(1 - \frac{1}{n}\right)^{n+1}}{\sqrt{(1-\frac{1}{n})^2 + \frac{2}{rn} - \frac{1}{rn^2}}};$$

rn being supposed a very great number, this function is reduced quite nearly to this very simple form

$$\left(\frac{n-1}{n}\right)^n.$$

This is therefore the expression approached more and more by the probability that none of the balls of the urn will come forth at its rank, when there is a great number of balls. The hyperbolic logarithm of this expression being

$$-1 - \frac{1}{2n} - \frac{1}{3n^2} - \cdots,$$

one sees that it always goes on increasing in measure as n increases; that it is null, when $n = 1$, and that it becomes $\frac{1}{c}$, when n is infinity, c being always the number of which the hyperbolic logarithm is unity.

We imagine now a number i of urns each containing the number n of balls, all of different colors, and that one draws successively all the balls in each urn. One can, by the preceding reasonings, determine the probability that one or more balls of the same color will come forth at the same rank in the i drawings. In reality, we suppose that the ranks of the colors are settled after the complete drawing of the first urn, and we consider first the first color; we suppose that it comes forth the first in the drawings of the $i - 1$ other urns. The total number of combinations of the $n - 1$ other colors in each urn is, by having regard for their situation among themselves, $1.2.3\dots(n - 1)$; thus the total number of these combinations relative to $i - 1$ urns is $[1.2.3\dots(n - 1)]^{i-1}$; this is the number of cases in which the first color is drawn the first altogether from all these urns, and, as there are n colors, one will have

$$n[1.2.3\dots(n-1)]^{i-1}$$

for the number of cases in which one color at least will arrive at its rank in the drawings from the $i - 1$ urns. But there are in this number some repeated cases; thus the case where two colors arrive at their rank in these drawings are contained twice in this number; it is necessary therefore to subtract them from it. The number of these cases is, by that which precedes,

$$\frac{n(n-1)}{1.2}[1.2.3\dots(n-2)]^{i-1};$$

by subtracting it from the preceding number, one will have the function

$$n[1.2.3\dots(n-1)]^{i-1} - \frac{n(n-1)}{1.2}[1.2.3\dots(n-2)]^{i-1}.$$

But this function contains itself some repeated cases. By continuing to exclude from them as one has made above relatively to a single urn, by dividing next the final function by the number of all possible cases, and which is here $(1.2.3\dots n)^{i-1}$, one will have, for the probability that one of the $n - 1$ colors at least will come forth at its rank in the $i - 1$ drawings which follow the first,

$$\frac{1}{n^{i-2}} - \frac{1}{1.2[n(n-1)]^{i-2}} + \frac{1}{1.2.3[n(n-1)(n-2)]^{i-2}} - \cdots,$$

an expression in which it is necessary to take as many terms as there are units in

n. This expression is therefore the probability that at least one of the colors will come forth at the same rank in the drawings of i urns.