CHAPTER VI

DE LA PROBABILITÉ DES CAUSES ET DES ÉVÈNEMENTS FUTURS, TIRÉE DES ÉVÈNEMENTS OBSERVÉS

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Théorie Analytique des Probabilités §§26–33, pp. 370–409

ON THE PROBABILITY OF CAUSES AND OF FUTURE EVENTS, DRAWN FROM OBSERVED EVENTS

An observed event being composed of simple events of the same kind and of which the possibility is unknown, to determine the probability that this possibility is comprehended within some given limits. Expression of this probability. Formula in order to determine it by a very convergent series, when the observed event is composed of a great number o these simple events. Extension of this formula to the case where the observed event is composed of many different kinds of simple events Nº 26.

Application of these formulas to the following problems: *Two players A and B play together to this condition that the one who out of three trials will have won two of them will win the set, the third trial not being played as useless, if the same player wins the first two trials. Out of a great number n of won sets, A has won the number i of them; one demands the probability that his skill, respectively to player B, is comprehended within some given limits.*

One demands the probability that the number of trials played is comprehended within some determined limits. Finally, this last number being supposed known, one demands the probability that the number of sets is comprehended within some given limits.

Solutions of these diverse problems. Nº 27.

Application of the formulas of nº 26 to the births observed in the principal places of Europe. Everywhere the number of births of boys is superior to the one of the births of girls. *To determine the probability that there exists a constant cause of this superiority, according to the births observed in a given place*. Solution of the problem. This probability for Paris differs excessively little from certitude. Nº 28.

At Paris, the ratio of the baptisms of the boys to those of the girls is $\frac{25}{24}$, while at London this ratio is $\frac{19}{18}$. *To determine the probability that there exists a constant cause of this difference.* Solution of the problem. This probability is very great. Probable conjecture with respect to this cause. Nº 29.

Research on the probability of the results based on the Tables of mortality or of assurance, constructed out of a great number of observations.

Supposing that, out of a great number p of individuals of age A, one has observed that there exists of them q at age $A+a$, r at age $A+a+a'$, ..., to determine the probability that, out of a great number p' of individuals of the same age A, there will exist of them $\frac{p'q}{p} \pm z$ at age $A+a$, $\frac{p'r}{p} \pm z'$ at age $A+a+a'$, ... Solution of the problem. There results from it that by increasing the number p one approaches without ceasing the true law of mortality, with which the results of the observations would coincide, if p was infinite. N° 30.

To evaluate, by means of annual births, the population of a vast empire. Solution of the problem. Application to France. Probability that the error of this evaluation will be comprehended within some given limits. N° 31.

Expression of the probability of a future event, drawn from an observed event. When the future event is composed of a number of simple events, much smaller than the one of the simple events which enter into the observed event, one is able, without sensible error, to determine the possibility of the future event, by supposing to each simple event the possibility which renders the observed event most probable. N° 32.

From the epoch when one has distinguished at Paris, out of the registers, the births of each sex, one has observed that the number of masculine births surpasses the one of the feminine births; to determine the probability that this annual superiority will be maintained within a given interval of time, for example, in the space of a century. N° 33.

26. The probability of the greater part of simple events is unknown: by considering it *a priori*, it appears to us susceptible of all the values comprehended between zero and unity; but, if one has observed a result composed of many of these events, the manner by which they enter there renders some of these values more probable than the others. Thus, in measure as the observed result is composed by the development of the simple events, their true possibility is made more and more known, and it becomes more and more probable that it falls within some limits which, being tightened without ceasing, would end by coinciding, if the number of simple events became infinite. In order to determine the laws according to which this possibility is discovered, we will name it x. The theory exposed in the preceding Chapters will give the probability of the observed result, in a function of x. Let y be this function; if one considers the different values of x as so many causes of this result, the probability of x will be, by the third principal of n° 1, equal to a fraction of which the numerator is y, and of which the denominator is the sum of all the values of y; by multiplying therefore the numerator and the denominator of this fraction by dx, this probability will be

$$\frac{y\,dx}{\int y\,dx},$$

the integral of the denominator being taken from $x = 0$ to $x = 1$. The probability that the value of x is comprehended within the limits $x = \theta$ and $x = \theta'$ is consequently equal to

$$\frac{y\,dx}{\int y\,dx}, \tag{1}$$

the integral of the numerator being taken from $x = \theta$ to $x = \theta'$, and that of the denominator being taken from $x = 0$ to $x = 1$.

The most probable value of x is that which renders y a maximum. We will designate it by a. If at the limits of x, y is null, then each value of y has an equal value corresponding on the other side of the maximum.

When the values of x, considered independently of the observed result, are not equally possible, by naming z the function of x which expresses their probability, it is easy to see, by that which has been said in Chapter I of this Book, that by changing in formula (1), y into yz, one will have the probability that the value of x is comprehended within the limits $x = \theta$ and $x = \theta'$. This reverts to supposing all the values of x equally possible *a priori*, and by considering the observed result as being formed of two independent results, of which the probabilities are y and z. One is able to restore thus all the cases to the one where one supposes *a priori*, before the event, an equal possibility to the different values of x, and, by this reason, we will adopt this hypothesis in that which will follow.

We have given in n$^{\text{os}}$ 22 and the following of Book I the formulas necessary in order to determine, by some convergent approximations, the integrals of the numerator and of the denominator of formula (1), when the simple events of which the observed event is composed are repeated a very great number of times; because then y has for factors some functions of x raised to very great powers. We will, by means of these formulas, determine the law of probability of the values of x, in measure as they deviate from the value a, the most probable, or which renders y a maximum. For that, we resume formula (*c*) of n$^{\text{o}}$ 27 of Book I,

$$(2)\qquad \begin{cases} \displaystyle\int y\,dx = Y\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2} + \frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right)\int dt\,c^{-t^2} \\ \displaystyle\qquad + \frac{Y}{2}c^{-T^2}\left[\frac{dU^2}{dx} - T\frac{d^2U^3}{1.2\,dx^2} + (T^2+1)\frac{d^3U^4}{1.2.3\,dx^3} - \cdots\right] \\ \displaystyle\qquad - \frac{Y}{2}c^{-T^2}\left[\frac{dU^2}{dx} + T'\frac{d^2U^3}{1.2\,dx^2} + (T'^2+1)\frac{d^3U^4}{1.2.3\,dx^3} + \cdots\right]; \end{cases}$$

ν is equal to $\frac{x-a}{\sqrt{\log Y - \log y}}$, and U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}$, $\ldots$ are that which ν, $\frac{d\nu^2}{dx}$, $\frac{d^2\nu^3}{dx^2}$, $\ldots$, become when one changes, after the differentiations, x into a, a being the value of x which renders y a maximum: T is equal to that which the function $\sqrt{\log Y - \log y}$ becomes, when one changes x into $a-\theta$ in y, and T' is that which the same function becomes, when one changes x into $a+\theta'$. The preceding expression of $\int y\,dx$ gives the value of this integral, within the limits $x = a-\theta$ and $x = a+\theta'$, the integral $\int dt\,c^{-t^2}$ being taken from $t=-T$ to $t=T'$.

Most often, at the limits of the integral $\int y\,dx$, extended from $x=0$ to $x=1$, y is null; now, when y is not null, it becomes so small at these limits, that one is able to suppose it null. Then, one is able to make at these limits T and T' infinite, this which gives for the integral $\int y\,dx$, extended from $x=0$ to $x=1$,

$$\int y\,dx = Y\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2} + \frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right)\sqrt{\pi};$$

thus the probability that the value of x is comprehended within the limits $x=a-\theta$ and $x=a+\theta'$ is equal to

$$(3)\qquad \frac{\int dt\,c^{-t^2}}{\sqrt{\pi}} + \frac{\begin{Bmatrix} \displaystyle\frac{1}{2}c^{-T^2}\left[\frac{dU^2}{dx} - T\frac{d^2U^3}{1.2\,dx^2} + (T^2+1)\frac{d^3U^4}{1.2.3\,dx^3} - \cdots\right] \\ \displaystyle -\frac{Y}{2}c^{-T^2}\left[\frac{dU^2}{dx} + T'\frac{d^2U^3}{1.2\,dx^2} + (T'^2+1)\frac{d^3U^4}{1.2.3\,dx^3} + \cdots\right] \end{Bmatrix}}{\left(U + \frac{1}{2}\frac{d^2U^3}{1.2\,dx^2} + \frac{1.3}{2^2}\frac{d^4U^5}{1.2.3.4\,dx^4} + \cdots\right)\sqrt{\pi}}$$

One sees, by n° 23 of Book I, that, in the case where y has for factors some functions of x elevated to great powers of order $\frac{1}{\alpha}$, α being an extremely small fraction, then U is most often of order $\sqrt{\alpha}$, so that its successive differences; U, $\frac{dU^2}{dx}$, $\frac{d^2U^3}{dx^2}, \ldots$ are respectively of the orders $\sqrt{\alpha}$, α, $\alpha^{\frac{3}{2}}$, $\ldots$; whence it follows that the convergence of the series of formula (3) requires that T and T' are not of an order superior to $\frac{1}{\sqrt{\alpha}}$.

If one supposes $\theta = \theta'$, then one has very nearly $T = T'$, and formula (3) is reduced, by neglecting the terms of order α, in the integral $\frac{\int dt\, c^{-t^2}}{\sqrt{\pi}}$, taken from $t = -T$ to $t = T'$; this which reverts, by neglecting the square of the difference ${T'}^2 - T^2$, by doubling the preceding integral and by taking it from t null to

$$t = \sqrt{\frac{T^2 + {T'}^2}{2}}.$$

Now one has

$$T^2 = \log Y - \log y,$$

and one is able to suppose

$$\log y = \frac{1}{\alpha} \log \phi,$$

ϕ being a function of x or of $\alpha - \theta$, which no longer contains factors raised to great powers. By naming therefore Φ, $\frac{d\Phi}{dx}$, $\frac{d^2\Phi}{dx^2}$, ... that which ϕ, $\frac{d\phi}{dx}$, $\frac{d^2\phi}{dx^2}$, ... become, when θ is null, by observing next that the condition of Y or Φ a maximum gives $\frac{d\Phi}{dx} = 0$, one will have

$$\alpha T^2 = -\theta^2 \frac{d^2\Phi}{2\Phi\, dx^2} + \theta^3 \frac{d^3\Phi}{6\Phi\, dx^3} - \frac{\theta^4}{8}\left[\frac{d^4\Phi}{3\Phi\, dx^4} - \left(\frac{d^2\Phi}{\Phi\, dx^2}\right)^4\right] + \cdots$$

By changing θ into $-\theta$, one will have the value of $\alpha {T'}^2$; one will have therefore, by neglecting the terms of order α^2,

$$\frac{\alpha(T^2 + {T'}^2)}{2} = -\theta^2 \frac{d^2\Phi}{2\Phi\, dx^2};$$

hence,

$$\sqrt{\frac{T^2 + {T'}^2}{2}} = \frac{\theta}{\sqrt{\alpha}} \sqrt{-\frac{d^2\Phi}{2\Phi\, dx^2}};$$

We make

$$k = \sqrt{-\frac{d^2\Phi}{2\Phi\,dx^2}} = \sqrt{-\frac{\alpha d^2 Y}{2Y dx^2}},$$
$$\theta = \frac{t\sqrt{\alpha}}{k};$$

the probability that the value of x is comprehended within the limits $a \pm \frac{t\sqrt{\alpha}}{k}$ will be

$$\frac{2\int dt\, c^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = 0$, and being able to be obtained in a quite near manner from the formulas of nº 27 from Book I.

There results from this expression that the most probable value of x is a, or that which renders the observed event the most probable, and that by multiplying to infinity the simple events of which the observed event is composed, one is able at the same time to narrow the limits $a \pm \frac{t\sqrt{\alpha}}{k}$, and to increase the probability that the value of x will fall between these limits; so that at infinity, this interval becomes null, and the probability is confounded with certitude.

If the observed event depends on simple events of two different kinds, by naming x and x' the possibilities of these two kinds of events, one will see, by the preceding reasonings, that, y being then the probability of the composite event, the fraction

$$\frac{y\,dx\,dx'}{\int\int y\,dx\,dx'} \tag{4}$$

will be the probability of the simultaneous values of x and of x', the integrals of the denominator being taken from $x = 0$ to $x = 1$, and from $x' = 0$ to $x' = 1$. By naming a and a' the values of x and x' which render y a maximum, and making $x = a + \theta$, $x' = a' + \theta'$, one will find, by the analysis of nº 27 from Book I, that if one supposes

$$\frac{\theta}{\sqrt{2Y}}\sqrt{-\frac{\partial^2 Y}{\partial x^2}} - \theta' \frac{\frac{\partial^2 Y}{\partial x \partial x'}}{2Y}\sqrt{\frac{-2Y}{\frac{\partial^2 Y}{\partial x^2}}} = t,$$

$$\frac{\theta'}{\sqrt{-2Y\frac{\partial^2 Y}{\partial x^2}}}\sqrt{\frac{\partial^2 Y}{\partial x^2}\frac{\partial^2 Y}{\partial x'^2} - \left(\frac{\partial^2 Y}{\partial x \partial x'}\right)^2} = t',$$

the fraction (4) taking this form

$$\frac{dt\, dt' c^{-t^2-t'^2}}{\int\int dt\, dt' c^{-t^2-t'^2}}.$$

The integrals of the denominator must be taken from $t = -\infty$ to $t = \infty$, and from $t' = -\infty$ to $t' = \infty$; because the integrals relative to x and x' of the fraction (4) being taken from $x = 0$ and $x' = 0$ to x and x' equal to unity, and at these limits, the values of θ and of θ' being $-a$ and $1 - a$, $-a'$ and $1 - a'$, the limits of t and of t' are equal to these last limits multiplied by some quantities of order $\frac{1}{\sqrt{\alpha}}$: thus the exponential $c^{-t^2-t'^2}$ is excessively small at these limits, and one is able, without sensible error, to extend the integrals of the denominator of the preceding fraction to the positive and negative infinite values of the variables t and t'. This denominator becomes thus equal to π; and the probability that the values of θ' and of θ are comprehended within the limits

$$\theta' = 0, \quad \theta' = \frac{t'\sqrt{-2Y\frac{\partial^2 Y}{\partial x^2}}}{\sqrt{\frac{\partial^2 Y}{\partial x^2}\frac{\partial^2 Y}{\partial x'^2} - \left(\frac{\partial^2 Y}{\partial x \partial x'}\right)^2}},$$

$$\theta = 0, \quad \theta = \frac{t\sqrt{2Y}}{\sqrt{-\frac{\partial^2 Y}{\partial x^2}}} + \frac{t'\frac{\partial^2 Y}{\partial x \partial x'}}{\sqrt{\frac{\partial^2 Y}{\partial x^2}\frac{\partial^2 Y}{\partial x'^2} - \left(\frac{\partial^2 Y}{\partial x \partial x'}\right)^2}}\sqrt{\frac{-2Y}{\frac{\partial^2 Y}{\partial x^2}}}$$

is equal to

$$\frac{1}{\pi}\int\int dt\, dt' c^{-t^2-t'^2},$$

the integrals being taken from t and t' nulls.

One sees by this formula that, in the case of two different kinds of simple events, the probability that their respective possibilities are those which render the composite event most probable becomes more and more great, and ends by being confounded with certitude; this which holds generally for any number whatsoever of different kinds of simple events, which enter into the observed event.

If one imagines an urn containing an infinity of balls of many different colors, and if after having drawn from it a great number n, p out of this number had been of the first color, q of the second, r of the third, etc.; by designating by x, x', x'', $\dots$ the respective probabilities to bring forth in a single drawing one of these colors, the probability of the observed event will be the term which has for factor $x^p x'^q x''^r \cdots,$ in the development of the polynomial

$$(x + x' + x'' + \cdots)^n,$$

where one has

$$\begin{aligned} x + x' + x'' + \cdots &= 1, \\ p + q + r + \cdots &= n; \end{aligned}$$

one will be able therefore to suppose here $y = x^p x'^q x''^r \cdots$, and then one has for the values of x, x', x'', $\dots$ which render the observed event the most probable

$$x = \frac{p}{n}, \quad x' = \frac{q}{n}, \quad x'' = \frac{r}{n}, \quad \dots$$

Thus the most probable values are proportionals to the numbers of the arrivals of the colors, and when the number n is a great number, the respective probabilities of the colors are very nearly equals to the numbers of times that they are arrived divided by the number of drawings.

27. In order to give an application of the preceding formula, we will consider the case where two players A and B play together with this condition, that the one who out of three trials will have won two of them wins the set, and we suppose that, out of a very great number n of sets, A has won a number i of them. By naming x the probability of A in order to win a trial, and consequently $1 - x$ the corresponding probability of B, the probability of A in order to win a set will be the sum of the first two terms of the binomial $(x + 1 - x)^3$, and the corresponding probability of B will be the sum of the last two terms. These probabilities are therefore $x^2(3 - 2x)$ and $(1 - x)^2(1 + 2x)$; thus the probability that, out of n sets, A will win i of them, and B, $n - i$, will be proportional to $x^{2i}(3 - 2x)^i(1 - x)^{2n-2i}(1 + 2x)^{n-i}$. By naming therefore y this function, and a the value of x which renders it a maximum, the probability that the value of x is

comprehended within the limits $a-\theta$ and $a+\theta$ will be

$$\frac{\int y\,dx}{\int y\,dx},$$

the integral of the numerator being taken from $x=a-\theta$ to $x=a+\theta$, and that of the denominator being taken from $x=0$ to $x=1$. If one makes

$$\frac{1}{n}=\alpha, \quad \frac{i}{n}=i',$$

one will have, by the preceding section,

$$\phi = x^{2i'}(3-2x)^{i'}(1-x)^{2-2i'}(1+2x)^{1-i'}.$$

The condition of the maximum of y or of ϕ gives $d\phi=0$; consequently, a being the value of x corresponding to this maximum, one will have

$$0=\frac{2i'}{a}-\frac{2i'}{3-2a}-\frac{2(1-i')}{1-a}+\frac{2(1-i')}{1+2a},$$

whence one draws

$$i'=a^2(3-2a), \quad 1-i'=(1-a)^2(1+2a);$$

next one has

$$\frac{-d^2\Phi}{2\Phi dx^2}=\frac{18}{(3-2a)(1+2a)}=k^2.$$

The probability that the value of x is comprehended within the limits $a\pm\frac{r}{\sqrt{n}}$ will be therefore, by the preceding section, equal to

$$\frac{6\sqrt{2}}{\sqrt{\pi(3-2a)(1+2a)}}\int dr\, c^{\frac{-18r^2}{(3-2a)(1+2a)}}.$$

One will see easily that this result agrees with the one that we have found in nº 16, by an analysis less direct that that here.

The set ends in two trials, if A or B wins the first two trials, the third trial not being played, because it becomes useless. Thus the numbers of sets won by one and the other of the players does not indicate the number of games played; but they indicate that this last number is contained within some given limits, with a probability that increases without ceasing, in measure as the sets multiply themselves. The research of this number and of this probability being very proper to clarify the preceding analysis, we will occupy ourselves with it.

The probability that A will win a set in two trials is x^2, x expressing, as above, his probability to win at each trial. The probability that he will win the set in three trials is $2x^2(1-x)$. The sum $x^2(3-2x)$ of these two probabilities is the probability that A will win the set. Thus, in order to have the probability that, out of i sets won by player A, s will be of two trials, it is necessary to raise to the power i the binomial

$$\frac{x^2}{x^2(3-2x)} + \frac{2x^2(1-x)}{x^2(3-2x)}$$

or

$$\frac{1}{3-2x} + \frac{2(1-x)}{3-2x},$$

and the term $i-s+1$ of the development of this power will be that probability which is thus equal to

$$\frac{1.2.3\ldots i.2^{i-s}(1-x)^{i-s}}{1.2.3\ldots s.1.2.3\ldots(i-s)(3-2x)^i}.$$

The greatest term of this development is, by nº 16, the one in which the exponents s and $i-s$ of the first and of the second term of the binomial are very nearly in the ratio of these terms, this which gives

$$s = \frac{i}{3-2x}.$$

We will name s' this quantity, and we will make

$$s = s' + l.$$

One will have, by nº 16,

$$\sqrt{\frac{i}{2s'\pi(i-s')}}dl\, c^{\frac{-il^2}{2s'(i-s')}}$$

for the probability of s, corresponding to the skill x of player A.

One will find similarly that, if one names z the number of the sets of two trials, won by player B, out of the number $n-i$ of sets that he has won, the most probable value of z will be $\frac{n-i}{1+2x}$, and that by designating by z' this quantity and making

$$z = z' + l',$$

the probability of z corresponding to x will be

$$\sqrt{\frac{n-i}{2z'(n-i-z')\pi}}dl'\, c^{\frac{-(n-i)l'^2}{2z'(n-i-z')}}.$$

The product of these two probabilities is therefore the probability corresponding to x, that the number of sets of two trials, won by player A, will be $s'+l$, while the number of sets of two trials, won by player B, will be $z'+l'$. Let

$$q = \frac{i}{2s'(i-s')}, \quad q' = \frac{n-i}{2z'(n-i-z')};$$

one will have, for this composite probability,

$$\frac{\sqrt{qq'}}{\pi}dl\, dl' c^{-ql^2 - q'l'^2}.$$

It is necessary to multiply this probability by that of x, which, as one has seen in the preceding section, is $\frac{y\,dx}{\int y\,dx}$; the product is

$$(\epsilon) \qquad \frac{\sqrt{qq'}}{\pi}\,\frac{y\,dx}{\int y\,dx}dl\, dl'\, c^{-ql^2 - q'l'^2}.$$

The integral of the denominator must be taken from $x = 0$ to $x = 1$, and by n° 27 of Book I, this integral is, very nearly,

$$Y\sqrt{\pi}\sqrt{\frac{-2Y}{\frac{d^2Y}{dx^2}}}.$$

If one names X the function

$$\sqrt{qq'}\, c^{-ql^2 - q'l'^2}$$

and if one designates by a' the value of x which renders Xy a maximum, and by

X' and Y' that which X and y become when one changes x into a', one will have, by the preceding section, by making $x = a' + \theta$,

$$ydx\sqrt{qq'}\, c^{-ql^2-q'l'^2} = Y'X'd\theta\, c^{\frac{\theta^2 d^2(X'Y')}{2X'Y'dx^2}}.$$

It is easy to see that a' differs from the value a of x which renders y a maximum, only by a quantity of order α, which we will designate by $f\alpha$; by substituting into Y, $a + f\alpha$ instead of a', in order to form Y', and developing with respect to the powers of α, one will see that $\frac{dY}{da}$ being null, because Y is the maximum of y, Y' differs from Y only by quantities of order α; thus one has, in the quantities near of an order inferior to the one that one conserves, and by observing that $\frac{dX'}{X'dx}$ and $\frac{d^2X'}{X'dx^2}$ are able to be neglected with respect to $\frac{dY'}{Y'dx}$,

$$\frac{d^2X'Y'}{2X'Y'dx^2} = \frac{d^2Y}{2Ydx^2};$$

the function (ϵ) becomes thence

$$(\epsilon') \qquad \frac{\sqrt{qq'}}{\pi\sqrt{\pi}}\sqrt{-\frac{d^2Y}{2Ydx^2}}\, dl\, dl'd\theta\, c^{-ql^2-q'l'^2+\frac{\theta^2d^2Y}{2Ydx^2}}.$$

One must, in this function, suppose $x = a$, this which gives, by substituting for i its value $na^2(3 - 2a)$,

$$q = \frac{3-2a}{4na^2(1-a)}, \quad q' = \frac{1+2a}{4na(1-a)^2}.$$

Next, x being equal to $a' + \theta$, it is equal to $a + f\alpha + \theta$; by neglecting therefore the quantities of order α, one will have

$$x = a + \theta.$$

Now the number of sets of two trials being

$$\frac{i}{3-2x} + \frac{n-i}{1+2x} + l + l',$$

this number will be

$$\frac{i}{3-2a} + \frac{n-i}{1+2a} + \left[\frac{2i}{(3-2a)^2} - \frac{2(n-i)}{(1+2a)^2}\right]\theta + l + l'.$$

We make

$$t = \left[\frac{2i}{(3-2a)^2} - \frac{2(n-i)}{(1+2a)^2}\right]\theta + l + l',$$

and we designate by q'' the quantity

$$-\frac{d^2Y}{2Y dx^2 \left[\frac{2i}{(3-2a)^2} - \frac{2(n-i)}{(1+2a)^2}\right]^2},$$

which, after all the reductions, is reduced to

$$\frac{9(3-2a)(1+2a)}{2n(1-2a)^2(3-2a+2a^2)^2};$$

the function (ϵ') will become

$$(\epsilon'') \qquad \frac{\sqrt{qq'q''}}{\pi\sqrt{\pi}}\, dt\, dl\, dl'\, d\theta\, c^{-ql^2 - q'l'^2 - q''(t-l-l')^2}.$$

By integrating from $l = -\infty$ to $l = \infty$, and from $l' = -\infty$ to $l' = \infty$, one will have the probability that the number of sets of two trials will be equal to

$$\frac{i}{3-2a} + \frac{n-i}{1+2a} + t;$$

now one has

$$\int dl\, c^{-ql^2 - q'l'^2 - q''(t-l-l')^2} = \int dl\, c^{-\frac{qq''}{q+q''}(t-l')^2 - q'l'^2 - (q+q'')\left[l - \frac{q''}{q+q''}(t-l')\right]^2}.$$

This last integral, taken from $l = -\infty$ to $l = \infty$, is, by that which precedes,

$$\frac{\sqrt{\pi}}{\sqrt{q+q''}} c^{-\frac{qq''}{q+q''}(t-l')^2 - q'l'^2}.$$

By multiplying it by dl' and by putting it under this form

$$\frac{\sqrt{\pi}\, dl'}{\sqrt{q+q''}} c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''} - \frac{qq'+qq''+q'q''}{q+q''}\left(l' - \frac{qq''t}{qq'+qq''+q'q''}\right)^2},$$

and integrating from $l' = -\infty$ to $l' = \infty$, one will have

$$\frac{\pi}{\sqrt{qq' + qq'' + q'q''}}\, c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''}}$$

The function (ϵ'') integrated with respect to l and l', within the positive and negative infinite limits of these variables, becomes thus

$$\frac{1}{\sqrt{\pi}}\sqrt{\frac{qq'q''}{qq' + qq'' + q'q''}}\, c^{-\frac{qq'q''t^2}{qq'+qq''+q'q''}}.$$

Thus the probability that the number of sets of two trials will be comprehended within the limits

$$\frac{i}{3-2a} + \frac{n-i}{1+2a} \pm t = n[a^2 + (1-a)^2] \pm t$$

is equal to the double of the integral of the preceding differential, taken from t null. One must observe that q, q', q'' are of order $\frac{1}{n}$, so that the quantity $\frac{qq'q''}{qq'+qq''+q'q''}$ is of the same order. We represent it by $\frac{k'^2}{n}$, and we make $t = r\sqrt{n}$; one will have

$$(\epsilon''') \qquad \frac{2}{\sqrt{\pi}}\int k'dr\, c^{-k'^2r^2},$$

for the expression of the probability that the number of sets of two trials will be comprehended within the limits

$$n[a^2 + (1-a)^2] \pm r\sqrt{n},$$

the integral being taken from r null. The interval of these two limits is $2r\sqrt{n}$, and the ratio of this interval to the number n of sets is $\frac{2r}{\sqrt{n}}$. This ratio diminishes without ceasing in measure as n increases, and r is able to increase at the same time indefinitely, so that the preceding integral approaches indefinitely from unity.

The total number of trials is the triple of the number of sets of three trials, plus the double of the number of sets of two trials, or the triple of the total number n of sets, less the number of sets of two trials; it is therefore

$$2n(1 + a - a^2) \mp r\sqrt{n}.$$

The integral (ϵ''') is therefore the expression of the probability that the number of trials will be comprehended within these limits.

If, instead of knowing the number i of sets won by player A and the total number n of sets, one knows the number i and the total number of trials, the same analysis will be able to serve to determine the unknown number n of sets. For this, we designate by h the total number of trials; one will have, by that which precedes, the two equations

$$3n - \frac{i}{3-2a} - \frac{n-i}{1+2a} = h \pm r\sqrt{n},$$
$$\frac{i}{n} - \frac{i}{3-2a} = \frac{n-i}{1-a} - \frac{n-i}{1+2a}.$$

These equations give a and n in functions of $h \pm r\sqrt{n}$. We suppose

$$n = i\psi\left(\frac{h \pm r\sqrt{n}}{i}\right), \quad a = \Gamma\left(\frac{h \pm r\sqrt{n}}{i}\right);$$

one will have, by reducing into series,

$$n = i\psi\left(\frac{h}{i}\right) \pm ir\sqrt{n}\frac{d\psi\left(\frac{h}{i}\right)}{dh} + \cdots;$$

one will substitute into k', instead of n and of a, $i\psi\left(\frac{h}{i}\right)$ and $\Gamma\left(\frac{h}{i}\right)$: the integral (ϵ''') is then the probability that the number n of sets is comprehended within the limits

$$i\psi\left(\frac{h}{i}\right) \pm ir\sqrt{i\psi\left(\frac{h}{i}\right)}\frac{d\psi\left(\frac{h}{i}\right)}{dh}.$$

28. It is principally in the births that the preceding analysis is applicable, and one is able to deduce from it, not only for the human race, but for all the kinds of organized beings, some interesting results. Until here the observations of this kind have been made in great number only on the human race; we will submit the principals to the calculus.

We consider first the births observed at Paris, at London and in the realm of Naples. In the space of forty years elapsed from the commencement of 1745, an epoch where one has begun to distinguish at Paris, out of the registers, the births of two sexes, to the end of 1784, one has baptized in this capital 393386 boys and 377555 girls, the found infants being comprehended in this number: this gives nearly $\frac{25}{24}$ for the ratio of the baptisms of the boys to those of the girls.

In the space of eighty-five years elapsed from the commencement of 1664 to the end of 1758, there is born at London 737629 boys and 698958 girls, this which gives $\frac{19}{18}$ nearly, for the ratio of the births of boys to those of girls.

Finally, in the space of nine years elapsed, from the commencement of 1774 to the end of 1782, there is born in the realm of Naples, Sicily not included, 782352 boys and 746821 girls, this which gives $\frac{22}{21}$ for the ratio of the births of the boys to those of the girls.

The smallest of these numbers of births are related to Paris; besides it is in this city that the births of the boys and of the girls approach more to equality. For these two reasons, the probability that the possibility of the birth of a boy surpasses $\frac{1}{2}$ must be less than at London and in the realm of Naples. We determine numerically this probability.

We name p the number of masculine births observed at Paris, q the one of the feminine births, and x the possibility of a masculine birth, that is to say the probability that an infant who must be born will be a boy; $1-x$ will be the possibility of a feminine birth, and one will have the probability that, out of $p-q$ births, p will be masculine, and q will be feminine, equal to

$$\frac{1.2.3\ldots(p+q)}{1.2.3\ldots p.1.2.3\ldots q}x^p(1-x)^q,$$

By making therefore

$$y = x^p(1-x)^q$$

the probability that the value of x is comprehended within some given limits will by, by n° 26, equal to

$$\frac{\int y\,dx}{\int y\,dx},$$

the integral of the denominator being taken from $x=0$ to $x=1$, and that of the numerator being taken within the given limits. If one takes zero and $\frac{1}{2}$ for these limits, one will have the probability that the value of x not surpass $\frac{1}{2}$. The value which corresponds to the maximum of y is $\frac{p}{p+q}$, and, seeing the magnitude of the numbers p and q, the excess of $\frac{p}{p+q}$ over $\frac{1}{2}$ is too considerable in order to employ here formula (c) from n° 27 of Book I, in the approximation of the integral $\int y\,dx$, taken from $x = 0$ to $x = \frac{1}{2}$; it is necessary therefore, in this case, to make use of formula (A) from n° 22 of the same Book. Here one has

$$\nu = -\frac{y\,dx}{dy} = -\frac{x(1-x)}{p-(p-q)x};$$

the formula cited (A) gives thus, for the integral $\int y\ dx$ taken from $x = 0$ to $x = \frac{1}{2}$,

$$\frac{1}{2^{p+q+1}(p-q)}\left[1 - \frac{p+q}{(p-q)^2} + \cdots\right].$$

As for the integral $\int y\,dx$ taken from $x = 0$ to $x = 1$, one has, by n° 26,

$$\int y\,dx = Y\left(U + \frac{1}{2}\,\frac{d^2U^3}{1.2\,dx^2} + \cdots\right)\sqrt{\pi},$$

Y being that which y becomes at its maximum, or when one substitutes $\frac{p}{p+q}$ for x; ν is here equal to $\frac{x-\frac{p}{p+q}}{\sqrt{\log Y - \log y}}$, and U, $\frac{d^2U^3}{dx^2}$, ... are that which ν, $\frac{d^2\nu^3}{dx^2}$, ... become, when one makes, after the differentiations, $x = \frac{p}{p+q}$. One finds thus, for the integral $\int y\,dx$ taken from $x = 0$ to $x = 1$,

$$\int y\,dx = \frac{p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}\sqrt{2\pi}}{(p+q)^{p+q+\frac{3}{2}}}\left[1 + \frac{(p+q)^2 - 13pq}{12pq(p+q)} + \cdots\right];$$

the probability that the value of x does not surpass $\frac{1}{2}$ is therefore equal to

$$(o)\qquad \frac{(p+q)^{p+q+\frac{3}{2}}}{(p-q)\sqrt{\pi}\,2^{p+q+\frac{3}{2}}p^{p+\frac{1}{2}}q^{q+\frac{1}{2}}}\left[1 - \frac{p+q}{(p-q)^2} - \frac{(p+q)^2-13pq}{12pq(p+q)} - \cdots\right].$$

In order to apply some large numbers to this formula, it would be necessary to have the logarithms of p, q and $p-q$, with twelve decimals at least: one is able to supply it in this manner. One has

$$\log\left[\frac{\left(\frac{p+q}{2}\right)^{p+q}}{p^q q^q}\right] = -p\log\left(1 + \frac{p-q}{p+q}\right) - q\log\left(1 - \frac{p-q}{p+q}\right).$$

When the logarithms are hyperbolic, the second member of this equation, reduced to series, becomes

$$-(p+q)\left[\frac{\left(\frac{p-q}{p+q}\right)^2}{1.2}+\frac{\left(\frac{p-q}{p+q}\right)^4}{3.4}+\frac{\left(\frac{p-q}{p+q}\right)^6}{5.6}+\frac{\left(\frac{p-q}{p+q}\right)^8}{7.8}+\cdots\right]$$

One will have therefore, by this very convergent series, the hyperbolic logarithm of $\frac{(p+q)^{p+q}}{2^{p+q}p^pq^q}$. In multiplying it by $0,43429448$, one will convert it by tabular logarithm, and, by adding to it the tabular logarithm of $\frac{(p+q)^{\frac{3}{2}}}{2(p-q)\sqrt{2pq\pi}}$, one will have the tabular logarithm of the factor which multiplies series (*o*). If one names $\frac{1}{\mu}$ this factor and if one makes

$$p = 393386, \quad q = 377555,$$

one finds by tabular logarithm

$$\log\mu = 72,2511780,$$

and the series (*o*) becomes

$$\frac{1}{\mu}(1 - 0,0030761 + \cdots).$$

This quantity of excessive smallness, subtracted from unity, will give the probability that at Paris the possibility of the births of the boys surpasses that of girls; whence one sees that one must regard this probability as being equal, at least, to that of the most authenticated historical facts.

If one applies formula (*o*) to the births observed in the principal cities of Europe, one finds that the superiority of the births of boys over the births of girls, observed everywhere from Naples to Petersburg, indicates a greater possibility of the births of boys, with a probability extremely near to certitude. This result appears therefore to be a general law, at least in Europe, and if, in some small cities, where one has observed only a little considerable number of births, nature seems to deviate from it, there is everywhere to believe that this deviation is only apparent, and that at length the observed births in these cities would offer, in being multiplied, a result similar to the one of the great cities. Many philosophers, deceived by these anomalies, have sought the cause of phenomena which are only the effect of chance; this which proves the necessity to make precede from parallel researches for that of the probability with which the observations indicate the phenomena of which one wishes to determine the cause. I take for example the small city of Vitteaux, in which, out of 415 observed births during five years, there is born 203 boys and 212 girls; p being here less than q, the natural order appears reversed. We see what is, after these observations, the probability that the facilities of the births of boys surpasses in this city that of the births of girls. This probability is $\frac{\int y\,dx}{\int y\,dx}$, the integral of the numerator being taken from $x = \frac{1}{2}$ to $x = 1$, and that of the denominator being taken from $x = 0$ to $x = 1$. Formula (*o*), which, subtracted from unity, gives this fraction, becomes here divergent; we will employ then formula (3) from nº 26, which is reduced very nearly to its first term $\frac{\int dt\, c^{-t^2}}{\sqrt{\pi}}$, the integral being taken from the value of t which corresponds to $x = \frac{1}{2}$ to the value of t which corresponds to $x = 1$. Now one has, by the section cited,

$$t^2 = \log Y - \log y,$$

y being $x^p(1-x)^q$, and Y being the value of y corresponding to the maximum of y, which holds when $x = \frac{p}{p+q}$; the value of t^2 which corresponds to $x = \frac{1}{2}$ is $-\log\left[\frac{\left(\frac{p+q}{2}\right)^{p+q}}{p^q q^q}\right]$, this logarithm being hyperbolic, and being given, by that which precedes, by a very convergent series. The value of t^2 which corresponds to $x = 1$ is $t^2 = \infty$; one has therefore thus the two limits of the integral $\int dt\, c^{-t^2}$, an integral which it will be easy to obtain by the formulas which we have given for this object. One finds thus the probability that at Vitteaux the facilities of the births of boys surpasses those of girls equal to $0,33$; the superiority of the facility of the births of girls is therefore indicated by these observations, with a probability equal to $0,67,$ a probability much too weak to balance the analogy which carried us to think that at Vitteaux, as in all the cities where one observed a considerable number of births, the possibility of the births of boys surpasses that of the births of girls.

29. One has seen at London the observed ratio of the births of boys to those of girls is equal to $\frac{19}{18}$, while at Paris the one of the baptisms of boys to those of girls is only $\frac{25}{24}$. This seems to indicate a constant cause of this difference. We determine the probability of this cause.

Let p and q be the numbers of baptisms of boys and girls made at Paris in the interval from the beginning of 1745 to the end of 1784; by designating by x the possibility of the baptism of a boy, and making, as in the preceding section,

$$y = x^p(1-x)^q,$$

the most probable value of x will be that which renders y a maximum: it is therefore $\frac{p}{p+q}$; by supposing next

$$x = \frac{p}{p+q} + \theta,$$

the probability of the value of θ will be, by n° 26, equal to

$$\frac{d\theta}{\sqrt{\pi}}\sqrt{\frac{(p+q)^3}{2pq}}c^{-\frac{(p+q)^3}{2pq}\theta^2}.$$

By designating by p', q' and θ' that which p, q and θ become for London, one will have

$$\frac{d\theta'}{\sqrt{\pi}}\sqrt{\frac{(p'+q')^3}{2p'q'}}c^{-\frac{(p'+q')^3}{2p'q'}\theta'^2}$$

for the probability of θ'; the product

$$\frac{d\theta\ d\theta'}{\pi}\sqrt{\frac{(p+q)^3(p'+q')^3}{4pqp'q'}}c^{-\frac{(p+q)^3}{2pq}\theta^2-\frac{(p'+q')^3}{2p'q'}\theta'^2}$$

of these two probabilities will be therefore the probability of the simultaneous existence of θ and θ'. We make

$$\frac{p'}{p'+q'} + \theta' = \frac{p}{p+q} + \theta + t;$$

the preceding differential function becomes

$$\frac{d\theta\ dt}{\pi}\sqrt{\frac{(p+q)^3(p'+q')^3}{4pqp'q'}}c^{-\frac{(p+q)^3}{2pq}\theta^2-\frac{(p'+q')^3}{2p'q'}\left[\theta+t-\frac{p'q-pq'}{(p+q)(p'+q')}\right]^2}.$$

By integrating it for all the possible values of θ and next for all the positive values of t, one will have the probability that the possibility of the baptisms of boys is greater at London than at Paris. The values of θ are able to be extended from θ equal to $-\frac{p}{p+q}$ to θ equal to $1-\frac{p}{p+q}$; but, when p and q are very great numbers, the factor $c^{-\frac{(p+q)^3}{2pq}\theta^2}$ is so small at these two limits that one is able to regard it as null; one is able therefore to extend the integral relative to θ, from $\theta = -\infty$ to $\theta = \infty$. One sees, for the same reason, that the integral relative to t is able to be extended from $t = 0$ to $t = \infty$. By following the process of nº 27 for these multiple integrations, one will find easily that, if one makes

$$k^2 = \frac{(p+q)^3(p'+q')^3}{2p'q'(p+q)^3 + 2pq(p'+q')^3},$$
$$h = \frac{p'q - pq'}{(p+q)(p'+q')},$$
$$\theta + \frac{2pqk^2}{(p+q)^3}(t-h) = t',$$

this which gives $d\theta = dt'$, the preceding differential, integrated first with respect to t' from $t' = -\infty$ to $t' = \infty$, and next from $t = 0$ to t infinity, will give

$$\int \frac{k\,dt}{\sqrt{\pi}} c^{-k^2(t-h)^2}$$

for the probability that at London the possibility of the baptisms of boys is greater than at Paris. If one makes

$$k(t-h) = t'',$$

this integral becomes

$$\int \frac{dt''}{\sqrt{\pi}} c^{-t''^2},$$

the integral being taken from $t'' = -kh$ to $t'' = \infty$, and it is clear that it is equal to

$$1 - \int \frac{dt''}{\sqrt{\pi}} c^{-t''^2},$$

the integral being taken from $t'' = kh$ to $t'' = \infty$. Thence it follows, by nº 27 of Book I, that, if one supposes

$$i^2 = \frac{p'q'(p+q)^3 + pq(p'+q')^3}{(p+q)(p'+q')(p'q-pq')^2},$$

the probability that the possibility of the baptisms of boys is greater at London than at Paris has for expression

$$(\mu)\qquad 1-\frac{ic^{-\frac{1}{2i^2}}}{\sqrt{2\pi}}\,\frac{1}{1+\frac{i^2}{1+\frac{2i^2}{1+\frac{3i^2}{1+\frac{4i^2}{1+\ddots}}}}}$$

By making in this formula

$$p = 393386, \quad q = 377555,$$
$$p' = 737629, \quad q' = 698958,$$

it becomes

$$1-\frac{1}{328269}.$$

There are therefore odds of 328268 against one that at London the possibility of the baptisms of boys is greater than at Paris. This probability approaches so much to certitude, that there is place to research the cause of this superiority.

Among the causes which are able to produce it, it has appeared to me that the baptisms of the found infants, who are part of the annual list of the baptisms at Paris, must have a sensible influence on the ratio of the baptisms of the boys to those of the girls, and that they must diminish this ratio, if, as it is natural to believe, the parents in the surrounding country, finding advantage to retain near to them the male infants, have sent from there to the hospice of the Enfants-Trouvés[1] of Paris, in a ratio less than the one of the births of the two sexes. It is this which the summary from the registers of this hospice has made me see with a very great probability. From the commencement of 1745 to the end of 1809, one has baptized 163499 boys and 159405 girls, a number of which the ratio is $\frac{39}{38}$, and differs too much from the ratio $\frac{25}{24}$ of the baptisms of the boys and the girls at Paris, in order to be attributed to simple chance.

30. We determine, according to the preceding principles, the probabilities of the results founded on the Tables of mortality or of assurance, constructed on a great number of observations. We suppose first that, with respect to a number p of individuals of a given age A, one has observed that there exists further the

[1] *Translator's note*: Foundling Hospital of Paris.

number q at the age $A+a$; one demands the probability that, out of p' individuals of age A, there will exist $q'+z$ of them at the age $A+a$, the ratio of p' and q' being the same as that of p to q.

Let x be the probability of an individual of age A, in order to survive to age $A+a$; the probability of the observed event is then the term of the binomial $x+(1-x)^p$ which has x^q for factor; this probability is therefore

$$\frac{1.2.3\dots p}{1.2.3\dots(p-q)1.2.3\dots q}x^q(1-x)^{p-q};$$

thus the probability of the value of x, taken from the observed event, is

$$\frac{x^q dx\,(1-x)^{p-q}}{\int x^q dx(1-x)^{p-q}},$$

the integral of the denominator being taken from $x=0$ to $x=1$.

The probability that, out of the p' individuals of age A, $q'+z$ will live to age $A+a$ is

$$\frac{1.2.3\dots p'}{1.2.3\dots(q'+z)1.2.3\dots(p'-q'-z)}x^{q'+z}(1-x)^{p'-q'-z}.$$

In multiplying this probability by the preceding probability of the value of x, the product integrated from $x=0$ to $x=1$ will be the probability of the existence of $q'+z$ persons at age $A+a$. By naming therefore P this probability, one will have

$$P=\frac{1.2.3\dots p'\int x^{q+q'+z}dx(1-x)^{p+p'-q-q'-z}}{1.2.3\dots(q'+z)1.2.3\dots(p'-q'-z)\int x^q dx(1-x)^{p-q}},$$

the integrals of the numerator and of the denominator being taken from $x=0$ to $x=1$. One has, by n° 28, very nearly,

$$\int x^{q+q'+z}dx(1-x)^{p+p'-q-q'-z}$$
$$= \sqrt{2\pi}\left[(q+q')\left(1+\frac{z}{q+q'}\right)\right]^{q+q'+z+\frac{1}{2}}$$
$$\times\ \frac{\left[(p+p'-q-q')\left(1-\frac{z}{p+p'-q-q'}\right)\right]^{p+p'-q-q'-z+\frac{1}{2}}}{(p+p')^{p+p'+\frac{3}{2}}},$$
$$\int x^q dx(1-x)^{p-q} = \sqrt{2\pi}\,\frac{q^{q+\frac{1}{2}}(p-q)^{p-q+\frac{1}{2}}}{p^{p+\frac{1}{2}}}.$$

Next, by n^o 33 of Book I, one has

$$1.2.3\ldots p' = p'^{p'+\frac{1}{2}}c^{-p'}\sqrt{2\pi},$$
$$1.2.3\ldots(q'+z) = q'^{q'+z+\frac{1}{2}}\left(1+\frac{z}{q'}\right)^{q'+z+\frac{1}{2}}c^{-q'-z}\sqrt{2\pi},$$
$$1.2.3\ldots(p'-q'-z) = (p'-q')^{p'-q'-z+\frac{1}{2}}\left(1-\frac{z}{p'-q'}\right)^{p'-q'-z+\frac{1}{2}}c^{-p'+q'+z}\sqrt{2\pi};$$

finally one has $q' = \frac{qp'}{p}$. This posed, one finds, after all the reductions,

$$P = \sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}\ \frac{\left(1+\frac{z}{q+q'}\right)^{q+q'+z+\frac{1}{2}}\left(1-\frac{z}{p+p'-q-q'}\right)^{p+p'-q-q'-z+\frac{1}{2}}}{\left(1+\frac{z}{q'}\right)^{q'+z+\frac{1}{2}}\left(1-\frac{z}{p'-q'}\right)^{p'-q'-z+\frac{1}{2}}}.$$

If one takes the hyperbolic logarithm of the second member of this equation, if one reduces this logarithm into series ordered with respect to the powers of z, and if one neglects the powers superior to the square, one will have, by passing again from the logarithm to the function,

$$P = \sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}\left[1+\frac{(2q-p)p^2z}{2qp'(p-q)(p+p')}\right]c^{\frac{-p^3z^2}{2qp'(p-q)(p+p')}}.$$

p, q, p' being supposed of very great numbers of order $\frac{1}{\alpha}$, the coefficient of z is very small of order z; the one of $-z^2$ is very small and of the same order. But, if one supposes $\frac{z}{p}$ of the order $\sqrt{\alpha}$, one will be able to neglect, in the preceding expression, the term depending on the first power of z, as very small of order

$\sqrt{\alpha}$. Moreover, this term is itself destroyed, when one has regard at the same time to the positive and negative values of z. By neglecting it therefore, one will have

$$2\sqrt{\frac{p^3}{qp'(p-q)(p+p')2\pi}}\int dz\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}$$

for the expression of the probability that, out of p' individuals of age A, the number of those who will arrive to age $A+a$ will be comprehended within the limits $q \pm z$, the integral being taken from z null.

We suppose now that one has found by observation that, out of p individuals of age A, q would live yet to age $A+a$, and r to age $A+a+a'$; one demands the probability that, out of p' individuals of the same age A, $\frac{qp'}{p}+z$ will live to age $A+a$, and $\frac{rp'}{p}+z'$ will live to age $A+a+a'$.

The probability that, out of p' individuals of age A, $\frac{qp'}{p}+z$ will live to age $A+a$ is, by that which precedes,

$$\sqrt{\frac{p^3}{2qp'(p-q)(p+p')2\pi}}\int dz\, c^{-\frac{p^3z^2}{2qp'(p-q)(p+p')}}.$$

One will have the probability that, out of $\frac{qp'}{p}+z$ individuals of age $A+a$, $\left(\frac{qp'}{p}+z\right)\frac{r}{q}+u$ will live to age $A+a+a'$, by changing in the preceding function p' into $\frac{qp'}{p}+z$, p into q, q into r and z into u; this which gives, by neglecting z with respect to $\frac{qp'}{p}$,

$$\sqrt{\frac{qp^2}{2rp'(q-r)(p+p')\pi}}\;\; c^{-\frac{qp^2u^2}{2rp'(p-r)(p+p')}}.$$

The product of these two probabilities is the probability of the simultaneous existence of z and of u. Now one has

$$\left(\frac{qp'}{p}+z\right)\frac{r}{q}+u=\frac{rp'}{p}+z',$$

this which gives

$$u = z' - \frac{rz}{q};$$

by making therefore

$$\beta^2 = \frac{p^3}{2qp'(p-q)(p+p')},$$
$$\beta'^2 = \frac{qp^2}{2rp'(q-r)(p+p')},$$

the probability P of the simultaneous existence of the values of z and of z' will be

$$P = \int \frac{\beta\, dz}{\sqrt{\pi}} \frac{\beta'\, dz'}{\sqrt{\pi}}\, c^{-\beta^2 z^2 - \beta'^2 \left(z' - \frac{rz}{q}\right)^2}.$$

By following this analysis, one finds generally that, if one makes

$$\beta''^2 = \frac{rp^2}{2sp'(r-s)(p+p')},$$
$$\beta'''^2 = \frac{sp^2}{2tp'(s-t)(p+p')},$$
$$\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots\dots$$

the probability P that, out of p' individuals of age A, the numbers of those who will live to ages $A+a$, $A+a+a'$, $A+a+a'+a''$, ... will be comprehended within the respective limits

$$\frac{qp'}{p},\ \frac{qp'}{p} + z; \quad \frac{rp'}{p},\ \frac{rp'}{p} + z'; \quad \frac{sp'}{p},\ \frac{sp'}{p} + z''; \quad \frac{tp'}{p},\ \frac{tp'}{p} + z'''; \quad \dots$$

is

$$P = \int \frac{\beta\, dz}{\sqrt{\pi}} \frac{\beta'\, dz'}{\sqrt{\pi}} \frac{\beta''\, dz''}{\sqrt{\pi}} \cdots c^{-\beta^2 z^2 - \beta'^2 \left(z' - \frac{rz}{q}\right)^2 - \beta''^2 \left(z'' - \frac{sz'}{r}\right)^2 - \cdots}.$$

One is able to estimate by this formula the respective probabilities of the numbers of a Table of mortality, constructed on a great number of observations. The manner to form these Tables is very simple. One takes out of the registers of births and of deaths a great number of infants who one follows during the course of their life, by determining how many there remain of them at the end of each year of their age, and one writes this number vis-à-vis dying each year. But, as in the first two or three years of life mortality if very rapid, it is necessary, for more exactitude, to indicate in this first age the number of the surviving at the end of each half-year. If the number p of infants were infinite, one would have thus some exact Tables which would represent the true law of mortality in the place and at the epoch of their formation. But the number of infants that one chose being finite, however great it be, the numbers of the Table are susceptible of errors. We represent by p', q', r', s', t', $\dots$ these diverse numbers. The true numbers, for a number p' of births, are $\frac{qp'}{p}$, $\frac{rp'}{p}$, $\frac{sp'}{p}$, $\frac{tp'}{p}$, $\dots$ If one makes $q' = \frac{qp'}{p} + z$, z will be the error of q'; similarly, if one supposes $r' = \frac{rp'}{p} + z'$, z' will be the error of r', and thus consecutively. The preceding expression of P is therefore the probability that the errors of q', r', s', $\dots$ are comprehended within the limits zero and z, zero and z', zero and z'', etc. The values of β, β', $\dots$ depend on p, q, r, $\dots$ which are unknowns; but the supposition of p infinite gives

$$\beta^2 = \frac{p^2}{2qp'(p-q)}.$$

One is able to substitute, without sensible error, $\frac{q'}{p'}$ instead of $\frac{q}{p}$, this which gives

$$\beta^2 = \frac{p'}{2q'(p'-q')}.$$

One will have in the same manner

$$\beta'^2 = \frac{q'}{2r'(q'-r')},$$
$$\beta''^2 = \frac{r'}{2s'(r'-s')},$$
$$\dots\dots\dots\dots\dots\dots$$

If one wishes to consider only the error of one of the number of the Table, such as s', then one will integrate the expression of P, relative to z''', z^{iv}, $\dots$, from the infinite negative values of these variables to their infinite positive values, and then one has

$$P = \int \frac{\beta\, dz}{\sqrt{\pi}} \frac{\beta'\, dz'}{\sqrt{\pi}} \frac{\beta''\, dz''}{\sqrt{\pi}} \;\; c^{-\beta^2 z^2 - \beta'^2\left(z' - \frac{r'z}{q}\right)^2 - \beta'''^2\left(z'' - \frac{s'z'}{r}\right)^2}.$$

The integrals relative to z and z' must be taken from their negative infinite values to their positive infinite values; one will find thus, by the process of which we have often made use for this kind of integration, that, if one supposes

$$\gamma^2 = \frac{p'}{2s'(p' - s')},$$

one will have

$$P = \int \frac{\gamma\, dz''}{\sqrt{\pi}}\, c^{-\gamma^2 z''^2}.$$

The probability that the error of any number from the Table will be comprehended within the limits zero and any quantity is therefore independent, either of the intermediate numbers, or of the subsequent numbers.

If one makes $\gamma z'' = t,$ one will have

$$\frac{z''}{s'} = t\sqrt{\frac{2(p' - s')}{p's'}},$$

and the probability P that the ratio of the error of the number s' from the Table to this number itself will be comprehended within the limits $\pm\, t\sqrt{\frac{2(p'-s')}{p's'}}$ is

$$P = 2\int \frac{dt}{\sqrt{\pi}}\, c^{-t^2},$$

the integral being taken from t null. One sees thus that, the value of t and consequently the probability P remaining the same, this ratio increases when s' diminishes; thus the numbers from the Table are so much less certain as they are more extended from the first p'. One sees further that this ratio diminishes in measure as p' increases, or in measure as one multiplies the observations; in a manner that one is able, by this multiplication, to diminish at the same time this ratio and to increase t, this ratio developing null when p' is infinite, and P becoming then equal to unity.

31. We apply the preceding analysis to the research on the population of a great empire. One of the simplest and most proper ways to determine this population is the observation of the annual births of which one is obliged to take account in order to determine the civil state of the infants. But this means supposes that one knows, very nearly, the ratio of the population to the annual births, a ratio that one obtains by making at many points of the empire the exact denumeration of the inhabitants, and by comparing it to the corresponding births observed during some consecutive years; one concludes from it next, by a simple proportion, the population of all the empire. The Government has well wished, at my prayer, to give some orders to have, with precision, these givens. In thirty departments, distributed over the surface of France, in a manner to outweigh the effects of the variety of climates, one has made a choice of the townships of which the mayors, by their zeal and their intelligence, would be able to furnish the most precise information. The exact denumeration of the inhabitants of these townships, for 22 September 1802, is totaled to 2037615 individuals. The summary of the births, of the marriages and of the deaths, from 22 September 1799 to 22 September 1802, has given, for these three years,

Births	Marriages	Deaths
110312 boys	46037	103659 males
105287 girls		99443 females.

The ratio of the births of boys to those of girls, that this summary presents, is the one of 22 to 21, and the marriages are to the births as 3 to 14; the ratio of the population to the annual births is $28,352845$. In supposing therefore the number of annual births in France equal to one million, this which deviates little from the truth, one will have, by multiplying by the preceding ratio, this last number, the population of France equal to 28352845 individuals. We see the error that one is able to fear in this evaluation.

For this, we imagine an urn which contains an infinity of white and black balls in an unknown ratio. We suppose next that having drawn at random a great number p of these balls, q have been white, and that, in a second drawing, out of an unknown number of extracted balls, there are q' white of them. In order to deduce from it this unknown number, one supposes its ratio to q', the same as the one of p to q, this which gives $\frac{pq'}{q}$ for this number. We seek the probability that the number of balls extracted at the second drawing is comprehended within the limits $\frac{pq'}{q} \pm z$. We name x the unknown ratio of the number of white balls to the total number of balls in the urn. The probability of the observed event in the first drawing will be expressed by the term which has for factor $x^q(1-x)^{p-q}$ in the development of the binomial $[x+(1-x)]^p$, whence it is easy to conclude, as in the preceding section, that the probability of x is

$$\frac{x^q dx\,(1-x)^{p-q}}{\int x^q dx\,(1-x)^{p-q}},$$

the integral of the denominator being taken from $x=0$ to $x=1$. We imagine that, in the second drawing, the total number of balls extracted is $\frac{pq'}{q}+z$; the probability of the observed number q' of white balls will be the term of the binomial $[x+(1-x)]^{\frac{pq'}{q}+z}$, which has for factor $x^{q'}(1-x)^{\frac{pq'}{q}+z-q'}$; this probability is therefore

$$\frac{1.2.3\ldots\left(\frac{pq'}{q}+z\right)}{1.2.3\ldots q'.1.2.3\ldots\left(\frac{pq'}{q}+z-q'\right)}x^{q'}(1-x)^{\frac{pq'}{q}+z-q'}.$$

By multiplying it by the preceding probability of x, by integrating the product from $x=0$ to $x=1$, and by dividing it by this same product multiplied by dz and integrated for all the positive and negative values of z, one will have the probability that the total number of balls extracted is $\frac{pq'}{q}+z$. One will find thus, by the analysis of the preceding section, this probability equal to

$$\sqrt{\frac{q^3}{2pq'(p-q)(q+q')\pi}}\;c^{-\frac{q^3z^2}{2pq'(p-q)(q+q')}}.$$

By naming this number P the probability that the number of balls extracted in the second drawing is comprehended within the limits $\frac{pq'}{q}\pm z$, one will have

$$P=1-2\int dz\sqrt{\frac{q^3}{2pq'(p-q)(q+q')\pi}}\;c^{-\frac{q^3z^2}{2pq'(p-q)(q+q')}},$$

the integral being taken from $z=z$ to z infinity.

Now, the number p of balls extracted in the first drawing is able to represent a denumeration, and the number q of white balls which are comprehended is able to express the number of women who, in this denumeration, must arrive mothers in the year, or the number of annual births, corresponding to the denumeration. Then q' expresses the number of annual births observed in all the empire, and whence one concludes the population $\frac{pq'}{q}$. In this case, the preceding value of P expresses the probability that this population is comprehended within the limits $\frac{pq'}{q} \pm z$.

We will suppose, conformably to the preceding givens,

$$p = 2037615, \qquad q = \frac{110313 + 105287}{3};$$

we will suppose next

$$q' = 1500000, \qquad z = 500000;$$

the preceding formula gives then

$$P = 1 - \frac{1}{1162}.$$

There are odds therefore around 1161 against one that in fixing at 42529267 the population corresponding to fifteen hundred thousand births, one will not be deceived by a half-million.

The difference between certitude and the probability P diminishes with a very great rapidity when z increases; it would be insensible if one supposed $z = 700000$.

32. We consider now the probability of future events, drawn from observed events, and we suppose that having observed an event composed of any number of simple events, one seeks the probability of a future result, composed of similar events.

We name x the probability of each simple event, y the corresponding probability of the observed result, and z the one of the future result; the probability of x will be, as one has seen,

$$\frac{y\,dx}{\int y\,dx},$$

the integral being taken from $x = 0$ to $x = 1$; $\frac{yz\,dx}{\int y\,dx}$ is therefore the probability of the future result, taken from the value of x, considered as cause of the simple event. Thus, by naming P the entire probability of the future event, one will have

$$P = \frac{yz\,dx}{\int y\,dx},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

We suppose, for example, that an event being arrived m times consecutively, one demands the probability that it will arrive the following n times. In this case, x being supposed to represent the possibility of the simple event, x^m will be that of the observed event, and x^n that of the future event, this which gives

$$y = x^m, \quad z = x^n,$$

whence one draws

$$P = \frac{m+1}{m+n+1}.$$

We suppose the observed event, composed of a very great number of simple events; let a be the value of x which renders y a maximum, and Y this maximum; let a' be the value of x which renders yz a maximum, and Y' and Z' that which y and z become at this maximum. One will have, by nº 27 of Book I, very nearly

$$\int y\,dx = \frac{Y^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{d^2Y}{dx^2}}},$$

$$\int yz\,dx = \frac{(Y'Z')^{\frac{3}{2}}\sqrt{2\pi}}{\sqrt{-\frac{d^2(Y'Z')}{dx^2}}}.$$

The observed result being composed of a very great number of simple events, we suppose that the future event is much less composite. The equation which gives the value of a' of x, corresponding to the maximum of yz, is

$$0 = \frac{dy}{y\,dx} + \frac{dz}{z\,dx};$$

$\frac{dy}{y\,dx}$ is a very great quantity, of order $\frac{1}{\alpha}$, and, since the future result is much less composite with respect to the observed result, $\frac{dz}{z\,dx}$ will be of a lesser order, which we will designate by $\frac{1}{\alpha^{1-\lambda}}$. Thus, a being the value of x which satisfies the equation $0 = \frac{dy}{y\,dx}$, the difference between a and a' will be very small of order α^λ, and one will be able to suppose

$$a' = a + \alpha^{\lambda}\mu.$$

This supposition gives

$$Y' = Y + \alpha^{\lambda}\mu \frac{dY}{dx} + \frac{\alpha^{2\lambda}\mu^2}{1.2}\frac{d^2Y}{dx^2} + \cdots$$

But one has $\frac{dY}{dx} = 0$, and it is easy to conclude from it that $\frac{d^nY}{Y\,dx^n}$ is of an order equal or less than $\frac{1}{\alpha^{\frac{n}{2}}}$; the term $\frac{\alpha^{n\lambda}\mu^n}{1.2.3\ldots n}\frac{d^nY}{Y\,dx^n}$ will be consequently more than order $\alpha^{n\left(\lambda-\frac{1}{2}\right)}$. Thus the convergence of the expression of Y' in series requires that λ surpass $\frac{1}{2}$, and in this case Y' differs from Y only by quantities of order $\alpha^{2\lambda-1}$.

If one names Z that which z becomes when one makes $x = a$, one will be assured in the same manner that Z' is able to be reduced to Z. Finally one will prove by a similar reasoning that $\frac{d^2(Y'Z')}{dx^2}$ is reduced to very nearly $Z\frac{d^2Y}{dx^2}$. By substituting these values into the expression of P, one will have

$$P = Z,$$

that is to say that one is able then to determine the probability of the future result, by supposing x equal to the value which renders the observed result most probable. But if it is necessary for that that the future result is rather little composite so that the exponents of the factors of z are of an order of magnitude smaller than the square root of the factors of y; alternately, the preceding supposition would expose some sensible errors.

If the future result is a function of the observed result, z will be a function of y, which we will represent by $\phi(y)$. The value of x which renders zy a maximum is, in this case, the same which renders y a maximum; thus one has $a' = a$, and if one designates $\frac{d\phi(y)}{dy}$ by $\phi'(y)$, the expression of P will become, by observing that $\frac{dY}{dx} = 0$,

$$P = \frac{\phi(Y)}{\sqrt{1 + \frac{Y\phi'(y)}{\phi(Y)}}}.$$

If $\phi(Y) = y^n$, so that the future event is n times the repetition of the observed event, one will have

$$P = \frac{Y^n}{\sqrt{n+1}}.$$

The probability P, calculated under the supposition that the possibility of the simple events is equal to that which renders the observed result most probable, is Y^n; one sees thus that the small errors which result from this supposition are accumulated at the rate of the simple events which enter into the future result, and become very sensible when these events are in great number.

33. Since 1745, an epoch where one has commenced to distinguish at Paris out of the registers the baptisms of boys from those of girls, one has constantly observed that the number of the first has been superior to the one of the second. We determine the probability that this superiority will be maintained during a given time, for example, in the space of a century.

Let p be the observed number of baptisms of boys, q the one of girls, $2n$ the number of annual baptisms, x the probability that the infant who will be born and be baptized will be a boy. By elevating $x + (1-x)$ to the power $2n$ and developing this power, one will have

$$x^{2n} + 2nx^{2n-1}(1-x) + \frac{2n(n-1)}{1.2}x^{2n-2}(1-x)^2 + \cdots$$

The sum of the n first terms of this development will be the probability that each year the number of baptisms of boys will surpass the one of the baptisms of girls. We name z this sum; z^i will be the probability that this superiority will be maintained during the number i of consecutive years; therefore, if one designates by P the entire probability that this will take place, one will have, by the preceding section,

$$P = \frac{\int x^p dx\, z^i (1-x)^q}{\int x^p dx\, (1-x)^q},$$

the integrals of the numerator and of the denominator being taken from $x = 0$ to $x = 1$.

If one names a the value of x which renders $x^p z^i (1-x)^q$ a maximum and if one designates by Z, $\frac{dZ}{dx}$, $\frac{d^2Z}{dx^2}$ this which z, $\frac{dz}{dx}$, $\frac{d^2z}{dx^2}$ becomes, when one changes x into a, one will have, by n° 26,

$$\int x^p dx\, z^i (1-x)^q = \frac{a^{p+1}(1-a)^{q+1} Z^i \sqrt{2\pi}}{\sqrt{p(1-a)^2 + qa^2 + ia^2(1-a)^2 \frac{dZ^2 - Zd^2Z}{Z^2dx^2}}}.$$

z being the sum of the first n terms of the function

$$x^{2n}\left[1 + 2n\frac{1-x}{x} + \frac{2n(2n-1)}{1.2}\,\frac{(1-x)^2}{x^2} + \cdots\right],$$

one has, by nº 37 of Book I,

$$z = \frac{\int \frac{u^{n-1}du}{(1+u)^{2n+1}}}{\int \frac{u^{n-1}du}{(1+u)^{2n+1}}},$$

the integral of the numerator being taken from $u = \frac{1-x}{x}$ to $u = \infty,$ and that of the denominator being take from $u = 0$ to $u = \infty$. Let $u = \frac{1-s}{s}$ be; this value of z will become

$$z = \frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s = 0$ to $s = x$, and that of the denominator taken from $s = 0$ to $s = 1$. Thence one draws

$$\frac{dz}{z\,dx} = \frac{x^n(1-x)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the denominator being taken from $x = 0$ to $s = x$. One will have next

$$\frac{d^2z}{z\,dx^2} = \frac{dz}{z\,dx} = \frac{n-(2n-1)x}{x(1-x)}.$$

By changing x into a in these expressions, one will have those of Z, $\frac{dZ}{Z\,dx}$, $\frac{d^2Z}{Z\,dx^2}$.

In order to determine a, we will observe that the condition of the maximum of $x^p z^i (1-x)^q$ gives

$$0 = \frac{p}{a} - \frac{q}{1-a} + i\frac{dZ}{Z\,dx},$$

whence one draws, by substituting for $\frac{dZ}{Z\,dx}$ its preceding value,

$$a = \frac{p}{p+q} + \frac{ia^{n+1}(1-a)^n}{(p+q)\int s^n ds\,(1-s)^{n-1}},$$

the integral of the denominator being taken from $x = 0$ to $s = a$. In order to conclude a from this equation, we will observe that the value of s which renders $s^n(1-s)^{n-1}$ a maximum is very nearly $\frac{1}{2}$, and consequently less than $\frac{p}{p+q}$, which itself is smaller than a. Thus, n being supposed a large number, one is able, without sensible error, to extend the integral of this expression of a, from $s = 0$ to $s = 1$, the term which depends on it being very small. This gives, by nº 28,

$$\int s^n ds\,(1-s)^{n-1} = \frac{n^{n+\frac{1}{2}}(n-1)^{n-\frac{1}{2}}\sqrt{2\pi}}{(2n-1)^{2n+\frac{1}{2}}} = \frac{\sqrt{\pi}}{2^{2n}\sqrt{n}}.$$

The equation which determines a becomes thus, quite nearly,

$$a = \frac{p}{p+q} + \frac{ia^{n+1}(1-a)^n 2^{2n}\sqrt{n}}{(p+q)\sqrt{\pi}}.$$

In order to resolve it, we will observe that a differs very little from $\frac{p}{p+q}$, so that, if one makes

$$a = \frac{p}{p+q} + \mu,$$

μ will be quite small, and one will have, in a most approximate manner,

(1) $$\mu = i\sqrt{n}\frac{p\left[1 - \left(\frac{p-q}{p+q}\right)^2\right]^n}{(p+q)^2\sqrt{\pi}} c^{-\frac{n\mu(p+q)(p-q)}{pq} - \frac{(p+q)^2 n\mu^2}{pq}};$$

one will have next, very nearly,

$$a^p(1-a)^q = \left(\frac{p}{p+q}\right)^p \left(\frac{q}{p+q}\right)^q c^{-\frac{(p+q)^3}{2pq}\mu^2}.$$

By substituting into the radical

$$\sqrt{p(1-a)^2+qa^2+ia^2(1-a)^2\frac{dZ^2-Zd^2Z}{Z^2dx^2}},$$

for a its value $\frac{p}{p+q}+\mu$, for $\frac{dZ}{Z\,dx}$ its value $\frac{(p+q)a-p}{ia(1-a)}$ or $\frac{(p+q)\mu}{ia(1-a)}$, and for $\frac{d^2Z}{Z\,dx^2}$ its value $\frac{dZ}{Z\,dx}\,\frac{n-(2n-1)a}{a(1-a)}$, this radical becomes very nearly

$$\sqrt{\frac{pq}{p+q}}\sqrt{1+\frac{(p+q)\mu}{pq}[n(p-q)-p]+\frac{(p+q)^2}{pq}\mu^2\left(2n+\frac{p+q}{i}\right)}.$$

Finally one has, by n° 28,

$$\int x^p dx\,(1-x)^q=\left(\frac{p}{p+q}\right)^p\left(\frac{q}{p+q}\right)^q\sqrt{\frac{pq}{p+q}}\,\frac{\sqrt{2\pi}}{p+q}.$$

This posed, the expression of P will become very nearly

(2) $$P=\frac{Z^i c^{-\frac{(p+q)^3}{2pq}\mu^2}}{\sqrt{1+\frac{(p+q)\mu}{pq}[n(p-q)-p]+\frac{(p+q)^2\mu^2}{pq}\left(2n+\frac{p+q}{i}\right)}}.$$

The concern is therefore no longer but to determine Z. One has

$$Z=\frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s=0$ to $s=a$, and that of the denominator being taken from $s=0$ to $s=1$. It is easy to conclude from it that one has

$$Z=1-\frac{\int s^n ds\,(1-s)^{n-1}}{\int s^n ds\,(1-s)^{n-1}},$$

the integral of the numerator being taken from $s=a$ to $s=1$ and that of the denominator being taken from $s=0$ to $s=1$; one will have thus, quite nearly, by n° 29,

(3) $$Z=1-\frac{\int dt\,c^{-t^2}}{\sqrt{\pi}},$$

the integral relative to t being taken from

$$t^2 = \frac{2n-1}{2n(n-1)}\left[\frac{n(p-q)}{p+q} - \frac{p}{p+q} + (2n-1)\mu\right]^2,$$

to $t^2 = \infty$.

In order to apply some numbers to these formulas, we will observe that, by that which precedes, in the interval from the commencement of 1745 to the end of 1784, one has by n° 28, relative to Paris,

$$p = 393386, \quad q = 377555.$$

By dividing by 40 the sum of these two numbers, one will have 19273,5 for the mean number of annual baptisms, this which gives $n = 9636,75$; we will suppose moreover $i = 100$. By means of these values one will determine that of μ by equation (1); one will determine next the value of Z by equation (3); finally equation (2) will give the value of P. One will find thus

$$P = 0,782.$$

There was therefore at the end of 1784, after these givens, nearly four against one to wager that, in the space of a century, the baptisms of boys at Paris will surpass, each year, those of girls.