

SUR LES NAISSANCES, LES MARIAGES ET LES MORTS

*At Paris, from 1771 to 1784; & in
the whole extent of France, during the
years 1781 & 1782.*

P.S. Laplace

Mém. Acad. R. Sci. Paris 1783, pp. 693-702.

The population is one of the most certain ways to judge the prosperity of an Empire; and the variations which it experiences, compared to the events which precede them are the most correct measure of the influence of the physical and moral causes on the happiness or on the unhappiness of the human race. It is therefore interesting in every regard to know the population of France, to follow the progress of it, & to have the law following which men are spreading over the surface of this great Realm. The researches keep too near to the Natural History of man, in order to be strangers to the Academy; they are too useful in order not to merit its attention. The Academy is determined by these consideration, to insert each year into its Mémoires, the list of births, of marriages & of deaths in the whole extent of France. A respectable magistrate by his light & by his zeal for the public good, & who since longtime occupies himself with success on researches on the population, has well wished to procure to himself all the information which it was able to desire on this matter; it is to him that we are indebted of the following lists. The first embraces the births, the marriages & the deaths in Paris, from 1771 to 1784; it serves to follow that which Mr. Morand has published in our Mémoires of 1771. The two other lists present the births, the marriages & the deaths in the whole extent of the Realm, during the years 1781 & 1872: it was desired that the sexes were distinguished, as they are in Paris, from 1745; but we must hope that convincing the Government of the importance of these results, will give them all the perfection of which they are susceptible.

Although the births are the source of the population; they are not sufficient however in order to determine it; we must yet know the mean duration of the existence of men in the place of their birth, whatever may be the causes which make them die there; because it is clear that in equality of births, a country will be so much more populous, as the men live a longer time: thus in the countries where the number of deaths were sensibly equal to that of the births, the population is very nearly constant, the number of years which express the mean duration of life, is the true ratio of the population to the annual births; it is the factor by which we must multiply this in order to have the population. The determination of this factor is the most delicate point & the most interesting of these researches: let us see how we can arrive to it.

The events of a like kind have some uniform and constant causes, but of which the action can be augmented or diminished by a thousand variable causes which produce the irregularities which we attribute to chance in the succession of events. These irregularities, by its offsetting the one by the others, would vanish in an infinite series of observations which would thus allow noticing only the result of the constant causes: but in a finite number of observations, they can deviate from this result, so much more as this number is less great. It is to these deviations which we must attribute the observed differences in the ratio of the population to the births, & there results from it the necessity to employ some great denumerations in order to determine this ratio. We will choose therefore a great number of parishes in all the provinces of the Realm, in order to have a mean among the small differences which the local causes can bring into the results: we will make an exact denumeration of their inhabitants in a given epoch; & by the revelation of the births during the ten years which precede this epoch, we will determine the corresponding number of annual births. By dividing by this number the one of the inhabitants, we will have the ratio of the population to the births, in a manner so much more precise, as the denumeration will be greater. As the number of annual births in France exceeds the one of the deaths, it is necessary, in order to establish an exact parity between the entire population of France & that of these parishes, to choose them in a manner that the total number of deaths is to the one of the births in the ratio which these two numbers have between themselves, relatively to all the Realm. If we take care to distinguish the sexes, we will have separately the population of the men, that of the women, & the duration of mean life of each of the two sexes, that which is interesting to know. A similar denumeration, made with care in diverse countries, & renewed in different centuries, would give the differences that the climate, the time & the governments can produce in the mean duration of the life of men.

The ratio of the population to the births, determined by the preceding method, can never be rigorously exact: by supposing in it even a rigorous precision, there would remain still on the population of France, the incertitude which is born of the action of the variable causes. The population of France, drawn from the annual births, is therefore only a probable result, & consequently susceptible to errors. It is to the analysis of chances to determine the probability of these errors, & to what point we must carry the denumeration, in order that it be very probable that they are contained within narrow limits. These researches depend on a new & yet little known theory, that of the probability of future events takes from observed events; they lead to some formulas of which the numerical calculation is impractical, because of the great numbers which we consider: but having given in this Volume & in the preceding, the principles necessary to resolve this kind of questions, & a general method to have in highly convergent series, the functions of great numbers; I have made application of it to the theory of the population deduced from births. The denumerations already made in France, & compared to the births, give very nearly 26 for the ratio of the population to the annual births; now if we take a mean among the births of the years 1781 & 1782, we have $973054\frac{1}{2}$ for the number of annual births in the whole extent of the Realm, containing in it Corsica; by multiplying therefore this number by 26, the population of the whole of France, will be 25299417 inhabitants. Now I find by my analysis, that in order to have a probability of a thousand to one, of not being deceived by a half-million in this evaluation of the population of France, it would be necessary that the denumeration which has served to determine the factor of 26, had been of 771469 inhabitants. If we would take $26\frac{1}{2}$ for the ratio of the population to the births, the number of the inhabitants of France will be 25785944; & in order to have the same probability of not being deceived by a half-million on this result, the factor $26\frac{1}{2}$ must be determined after a denumeration of 817219 inhabitants. It follows thence that if we wish to have for this object the precision which its importance requires, it is necessary to carry this denumeration to a million or twelve hundred thousand inhabitants. Here is the analysis which has lead me to this result.

We consider an urn which contains an infinity of white & black balls in an unknown ratio, & we suppose than in a first drawing we have extracted p white balls & q black balls; we suppose next that in a second drawing we have extracted q' black balls, but that we are ignorant of the number of white balls brought forth in this drawing; the mean which naturally presents itself in order to know this number in an approximate manner, is to suppose it with q' in the ratio

of p to q , that which gives $\frac{pq'}{q}$ for this number. We determine presently the probability that the true unknown number will be contained in the limits $\frac{pq'}{q} \cdot (1 - \varpi)$, & $\frac{pq'}{q} \cdot (1 + \varpi)$, or, that which returns to the same, that the error of the result $\frac{pq'}{q}$ will not surpass $\frac{pq'\varpi}{q}$.

For this, we name x the unknown ratio of the white balls to the total number of balls contained in the urn, & we designate by p' the unknown number of white balls brought forth in the second drawing; the probability of this drawing will be, by the known theory of chances,

$$\frac{1.2.3\dots(p' + q')}{1.2.3\dots p'.1.2.3\dots q'} \cdot x^{p'} \cdot (1 - x)^{q'}.$$

But p' being unknown, it is susceptible to all the values from $p' = 0$ to $p' = \infty$; these values are more or less probable, according as they render the second drawing more or less probable: we will have therefore the probability of p' , by dividing the preceding quantity, by the sum of all the values of this quantity, from $p' = 0$ to $p' = \infty$, that is by the infinite series,

$$(1 - x)^{q'} \cdot [1 + (q' + 1) \cdot x + \frac{(q' + 1)(q' + 2)}{1.2} \cdot x^2 + \&c.]$$

(See pages 428 & 429 of this Volume)¹. This series is equal to $\frac{1}{1-x}$; the probability of p' is therefore equal to

$$\frac{1.2.3\dots(p' + q')}{1.2.3\dots p'.1.2.3\dots q'} \cdot x^{p'} \cdot (1 - x)^{q'+1}.$$

This probability supposes that x is the ratio of the white balls to all the balls contained in the urn; but this ratio being unknown, we can make it vary from $x = 0$ to $x = 1$: these different values of x are more or less probable, according as they render the first drawing more or less probable; now, the probability of this drawing is

$$\frac{1.2.3\dots(p + q)}{1.2.3\dots p.1.2.3\dots q} \cdot x^p \cdot (1 - x)^q;$$

the probability of x will be therefore equal to $\frac{x^p \partial x (1-x)^q}{\int x^p \partial x (1-x)^q}$; the integral of the

¹ Suite du mémoire sur les approximations des Formules qui sont fonctions de très-grands nombres.(Note added by translator.)

denominator being taken from $x = 0$ to $x = 1$. (See page 430 of this volume.)² By multiplying this probability by that of p' , we will have the probability of p' , corresponding to the ratio x ; whence it follows that the entire probability of p' is equal to

$$\frac{1.2.3\dots(p' + q') \cdot \int x^{p+p'} \partial x \cdot (1-x)^{q+q'+1}}{1.2.3\dots p' \cdot 1.2.3\dots q' \cdot \int x^p \partial x \cdot (1-x)^q};$$

the integrals of the numerator & of the denominator being taken from $x = 0$ to $x = 1$.

The probability that p' is contained from $p' = 0$ to $p' = s$, will be, by virtue of the preceding formula,

$$\frac{\int x^p \partial x \cdot (1-x)^{q+q'+1} \cdot \left[1 + (q' + 1) \cdot x + \frac{(q'+1)(q'+2)\dots(q'+s)}{1.2.3\dots s} \cdot x^s \right]}{\int x^p \partial x \cdot (1-x)^q};$$

now, q' & s being supposed very large numbers, we will find by the analysis which I have given in the volume of 1782, page 60,³

$$\begin{aligned} & 1 + (q' + 1) \cdot x \dots + \frac{(q' + 1) \dots (q' + s)}{1.2.3\dots s} \cdot x^s \\ &= \frac{1}{(1-x)^{q'+1}} \cdot \frac{\int x'^s \partial x' (1-x')^q}{\int x'^s \partial x' (1-x')^q}; \end{aligned}$$

the integral of the numerator being taken from $x' = x$, to $x' = 1$, & that of the denominator being taken from $x' = 0$, to $x' = 1$: therefore the probability that p' is contained from $p' = 0$ to $p' = s$, is

$$\frac{\int \int x^p \partial x (1-x)^q \cdot x'^s \partial x' (1-x')^q}{\int \int x^p \partial x (1-x)^q \cdot x'^s \partial x' (1-x')^q};$$

the integrals of the numerator being taken from $x' = x$, to $x' = 1$; & from $x = 0$, to $x = 1$; those of the denominator being taken from x & x' null, to x & x' equal to unity. If we apply to this formula, the analysis that we have given

² Ibid. (Note added by translator.)

³ Mémoire sur les approximations des formules qui sont fonctions de très-grands nombres. (Note added by translator.)

pages 439 & following⁴ in this volume, we will find that if s is less & very little different from $\frac{pq'}{q}$, the preceding fraction will be very nearly equal to $\frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}}$, e being the number of which the hyperbolic logarithm is unity, π being the ratio of the semi-circumference to the radius, & the integral relative to t , being taken from $t = T$, to $t = \infty$, T being given by the equation

$$T^2 = \frac{\left(\frac{p}{p+q} - \frac{s}{s+q}\right)^2 \cdot (p+q)^3 \cdot (s+q')^3}{2sq'(p+q)^3 + 2pq(s+q')^3}.$$

We will find similarly that if s is greater than $\frac{pq'}{q}$, & if it differs little, the preceding fraction will be very nearly equal to $1 - \frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}}$, the integral being taken from $t = T$, to $t = \infty$. It follows thence that the probability that p' is contained between the two numbers s & s' of which the first is less, & the second greater than $\frac{pq'}{q}$, is equal to

$$1 - \frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}} - \frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}};$$

the first integral being taken from $t = T$, to $t = \infty$, & the second integral being taken from $t = T'$, to $t = \infty$, T & T' being given by the two equations

$$T^2 = \frac{\left(\frac{p}{p+q} - \frac{s}{s+q}\right)^2 \cdot (p+q)^3 \cdot (s+q')^3}{2sq'(p+q)^3 + 2pq(s+q')^3},$$

$$T'^2 = \frac{\left(\frac{p}{p+q} - \frac{s'}{s'+q'}\right)^2 \cdot (p+q)^3 \cdot (s'+q')^3}{2s'q'(p+q)^3 + 2pq(s'+q')^3}.$$

We suppose

$$s = \frac{pq'}{q}(1 - \varpi), \quad \& \quad s' = \frac{pq'}{q}(1 + \varpi),$$

ϖ being a very small fraction; if we neglect the quantities of order ϖ^3 , the two

⁴ Suite du mémoire sur les approximations des Formules qui sont fonctions de très-grands nombres. (Note added by translator.)

values of T^2 & T'^2 , will become equal between themselves & to $\frac{pq' \varpi^2}{2(p+q)(q+q')}$; thus by naming V^2 , this last quantity, & by designating by P the probability that the number p' will be contained within the limits $\frac{pq'}{q}(1 - \varpi)$, & $\frac{pq'}{q}(1 + \varpi)$, we will have

$$P = 1 - \frac{2 \int \partial t \cdot e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from $t = V$, to $t = \infty$. This quite simple expression for P , has the advantage of being exact to the quantities of order ϖ^4 ; because the terms of order ϖ^3 , which we have neglected, are destroyed among themselves in the quantity

$$1 - \frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}} - \frac{\int \partial t \cdot e^{-t^2}}{\sqrt{\pi}},$$

which we have found above for the expression of P .

It is easy to apply these results to the theory of the population deduced from births; because we can consider each annual birth as being represented by a black ball, & each existing individual as being represented by a white ball; the first drawing will be the denumeration in which we have observed that on q births, the number of the inhabitants is p ; & the second drawing will be the population of the whole of France of which the number q' of annual births is known, while the corresponding population p' is unknown: P will be in this case the probability that the population p' of France is contained within the limits $\frac{pq'}{q}(1 - \varpi)$, & $\frac{pq'}{q}(1 + \varpi)$; we will thus have this probability by a very simple formula.

It is easy to conclude from it the number to which p must be carried, in order to have a great probability that the error on the population p' of the whole of France will be very small. The research of this number becomes necessary, if we wish to make a new denumeration in order to determine the true factor by which we must multiply the annual births; thus we are going to enter into some details on this object.

For this, we will suppose

$$p = iq, \quad \frac{pq'}{q} \cdot \varpi = a;$$

we will have consequently $\varpi = \frac{a}{iq'}$, and the equation

$$V^2 = \frac{pq q' \cdot \varpi^2}{2(p+q) \cdot (q+q')}$$

will give

$$p = \frac{2i^2 \cdot (i+1) \cdot q'^2 \cdot V^2}{a^2 - 2i \cdot (i+1) \cdot q' \cdot V^2}.$$

This value of p supposes that we know a , q' , V & i . The value of a depends on the limits between which we suppose the error of the result $\frac{pq'}{q}$ is contained; we will make here $a = 500000$. The value of q' is given by the annual births of the whole extent of the Realm, & we have seen that $q' = 973054, 5$. The value of V depends on the probability P , that the population of France will be contained within the limits $\frac{pq'}{q} - a$ & $\frac{pq'}{q} + a$: we will suppose here that this probability is of a thousand to one, so that $P = \frac{1000}{1001}$; we will have thus

$$\frac{2 \int \partial t \cdot e^{-t^2}}{\sqrt{\pi}} = \frac{1}{1001}, \text{ or } \int \partial t \cdot e^{-t^2} = \frac{\sqrt{\pi}}{2002}.$$

The integral must be taken from $t = V$ to $t = \infty$, it is clear that this equation determines V , & we find $V^2 = 5, 415$. As for the number i , it depends on the ratio of p to q which results from the denumeration; but if the question is of the denumeration to make, this ratio is unknown; however the denumerations already made give very nearly $i = 26$; thus we are assured that the factor i deviates little from this number. We will suppose therefore successively $i = 25\frac{1}{2}$, $i = 26$, $i = 26\frac{1}{2}$, & we will have for the corresponding values of p ,

$$p = 727520, \quad p = 771469, \quad p = 817219,$$

that is, that in order to have a probability of one thousand against one, of not being deceived by one half-million in the evaluation of the population of France, it is necessary that the denumeration p , in the case where it gives the first factor, is of 727510 inhabitants; that it is 771469 inhabitants in the case of the second factor, & of 817219 inhabitants, if it leads to the third factor.

Thence I conclude that if we wish to have for this object, the probability that its importance requires, it is necessary to carry to a million or twelve hundred thousand inhabitants, the denumeration p which must determine the factor i .

© 2001 Richard J. Pulskamp
All rights reserved.