

TROISIÈME SUPPLÉMENT.

APPLICATION DES FORMULES GÉODÉSIQUES DE PROBABILITÉ A LA MÉRIDIDIENNE DE FRANCE

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Sequel to Second Supplement

1. The part of the meridian which extends from Perpignan to Formentera is supported on a base measured near Perpignan. Its length is around 466 km, and its last extremity is joined to the base of Perpignan by a chain of twenty-six triangles. One is able to fear that so great a length, which has not been verified at all by the measure of a second base toward its other extremity, is not susceptible to a sensible error arising from the errors of the twenty-six triangles employed to measure it. It is therefore interesting to determine the probability that this error not exceed 40^m or 50^m. Mr. Damoiseau, lieutenant-colonel of the Artillery, who has just gained the prize proposed by the Academy of Turin, on the return of the comet of 1759, has well wished, to my prayer, to apply to this part of the meridian my formulas of probability. Here the meridian does not cut all the triangles, as we have proposed for more simplicity; but it is easy to see that one is able to apply, to the angles formed by the elongations of the sides of the triangles with the meridian, that which I have said on the angles that these sides would form if they were cut by the meridian. Mr. Damoiseau has found thus that to depart from the latitude of the signal of Busgarach, a little more to the north than Perpignan, to Formentera, this which comprehends an arc of the meridian of about 466,006^m, and by taking for unity the base of Perpignan, one has (second Supplement, n^o 1)

$$\begin{aligned}
 p^2 - pq + q^2 + p_1^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \dots \\
 + p^{(25)2} - p^{(25)}q^{(25)} + q^{(25)2} = 48350,606.
 \end{aligned}$$

The probability that an error in the measure of this arc is contained within the limits $\pm s$ becomes, by the formulas of the same section,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to the value of t equal to

$$\frac{s}{2\theta} \frac{\sqrt{n+1}}{\sqrt{48350,606}},$$

$n+1$ being the number of triangles employed, and θ^2 being the sum of the squares of the errors observed in the sum of the three angles of each triangle; π is the ratio of the circumference to the diameter. By taking for unity the sexagesimal second, one finds

$$\theta^2 = 118,178$$

But, the number of triangles employed being only 26, it is preferable to determine by a great number of triangles this constant θ^2 which depends on the unknown law of the partial observations. For this, one has made use of the one hundred seven triangles which have served to measure the meridian from Dunkirk to Formentera. The set of the sums of observed errors of the three angles of each triangle is, in taking all of them positively, equal to 173,82. The sum of the squares of these errors is 445,217. By multiplying it by $\frac{26}{107}$, one will have, for the value of θ^2 ,

$$\theta^2 = 108,184.$$

This value, which differs little from the preceding, must be preferred. It is necessary to reduce θ into parts of the radius taken for unity, that which one will make by dividing it by the number of sexagesimal seconds that this radius renders. One will have thus

$$t = s 689,797;$$

s is a fraction of the base of Perpignan taken for unity. This base is 11706^m, 40. By supposing therefore the error of 60^m, one will have

$$t = \frac{60 \times 689,797}{11706,40}.$$

This put, one finds, for the probabilities of the errors of the arc of the meridian of which there is question are contained within the limits $\pm 60^m$, $\pm 50^m$, $\pm 40^m$,

the following fractions:

$$\frac{1743695}{1743696}, \quad \frac{32345}{32436}, \quad \frac{1164}{1165}.$$

There are odds one against one that the error falls within the limits $8^m, 0757$.¹

If the Earth were a spheroid of revolution and if the angles of all the triangles were exact, one would have exactly the inclination of the last side of the chain of the triangle on its meridian, by supposing given this inclination relative to the base. The probability that the error of the first of these inclinations, proceeding from the errors of the observed angles of the triangles, is comprised within the limits $\pm \frac{2}{3}\theta t$ is, by that which precedes,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null: these limits become, by substituting for θ its preceding value, $\pm t6''$, 8997, the seconds being sexagesimal. Thence it follows that there are odds one against one that the error falls within the limits $\pm 3''$, 2908. If the azimuthal observations were made with a great precision, one would determine by this means the probability that they indicate an eccentricity in the terrestrial parallels. If one measured, on the side of Spain, a base of verification equal to the base of Perpignan, and if one joined it by two triangles to the chain of triangles of the meridian, one finds, by the calculation, that there are odds one against one that the difference, between this base and its value concluded from the base of Perpignan, will not surpass a third of a meter: that is, to quite nearly, the difference of the measure of the base of Perpignan to its value concluded from the base of Melun.

One has seen, in the section cited, that, the angles of the triangles having been measured by means of the repeating circle, one is able to suppose the probability

¹ The comparison of this Memoir with the one which was published in the *Connaissance des Temps* for 1822 permits the identification of errors in this supplement. In the *Connaissance des Temps*, Laplace gives $\pm 8^m, 0937$. In reality, it would be $\pm 8^m, 0940$. We have $\frac{2}{\pi} \int_0^t e^{-t^2} dt = \frac{1}{2}$ for $t = 0, 476936$ thereby giving even odds. From the previous relation here,

$$t = \frac{s 689, 797}{11706, 40},$$

where $s = \pm 8^m, 0940$. The number given by Laplace differs slightly, without doubt, because the value of t , deduced from the formulas of Laplace, has not been calculated with as much precision as that employed later.

of an error x in the observed sum of the three angles of each triangle proportional to the exponential c^{-kx^2} , k being a constant. Thence it follows that the probability of this error is

$$\frac{dx \sqrt{k} c^{-kx^2}}{\sqrt{\pi}}.$$

By multiplying this differential by x and integrating from x null to x infinity, the double of this integral will be the mean of all the errors taken positively. By designating therefore by ϵ this mean error, one will have

$$\epsilon = \frac{1}{\sqrt{k\pi}}.$$

One will have the mean value of the squares of these errors by multiplying by x^2 the preceding differential and by integrating it from $x = -\infty$ to x infinity. By naming therefore ϵ' this value, one will have

$$\epsilon' = \frac{1}{2k};$$

thence one draws

$$\epsilon' = \frac{\epsilon^2 \pi}{2}.$$

One is able thus to obtain θ^2 by means of the errors, taken all in addition, the observed sums of the angles of each triangle. In the one hundred seven triangles of the meridian, the sum of the errors is 173, 82; one is able thus to take, for ϵ , $\frac{173,82}{107}$; this which gives, for $26\epsilon'$ or for θ^2 ,

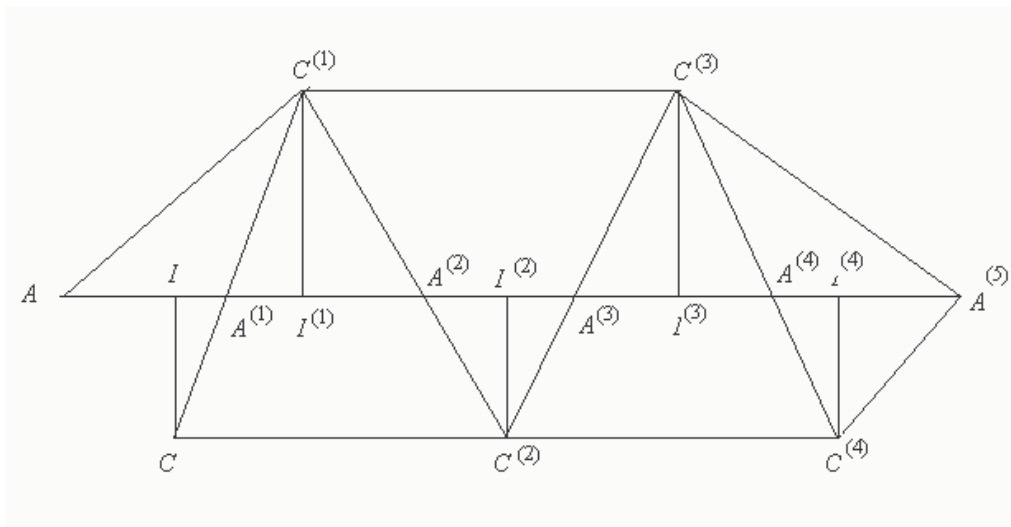
$$\theta^2 = 13\pi \left(\frac{173,82}{107} \right)^2 = 107,78;$$

this difference very near the value 108, 134 given by the sum of the squares of the errors of the observed sum of the angles of each of the one hundred seven triangles. This accord is remarkable.

One is able to appreciate the relative exactitude of the instruments of which one makes use in the geodesic observations, by the value of ϵ' concluded from a great number of triangles. This value, concluded from the one hundred seven triangles of the meridian, is $\frac{445,217}{107}$ or 4, 1609. The same value, concluded from

the forty-three triangles employed by La Condamine in the measure of the three degrees of the equator, is $\frac{1718}{43}$ or 39,953, and, consequently, nearly ten times greater than the preceding. The equally probable errors, relative to the instruments employed in these two operations, are proportionals to the square roots of the values of ϵ' . Thence it follows that the limits $\pm 8,0937$ m, between which we have just seen that there are odds one against one that the error of the arc measured from Perpignan to Formentera falls, would have been $\pm 25^m,022$ with the instruments employed by La Condamine. These limits would have surpassed $\pm 40^m$ with the instruments employed by La Caille and Cassini in their measure of the meridian. One sees thus how the introduction of the repeating circle in the geodesic operations has been advantageous.

2. In order to give a very simple example of the application of the geodesic formulas, I am going to consider the straight line $AA^{(5)}$, of which one has determined the length by a chain of triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$,...



I will suppose all these triangles equal and isosceles, and such that their bases $CC^{(2)}$, $C^{(1)}C^{(3)}$,... are parallels to the line $AA^{(5)}$. One will have, by lowering onto this line the perpendiculars CI , $C^{(1)}I^{(1)}$,...

$$\begin{aligned}
II^{(1)} &= CC^{(1)} \cos A^{(1)}, \\
C^{(1)}C^{(2)} &= \frac{CC^{(1)} \sin C^{(1)} CC^{(2)}}{\sin C^{(1)} C^{(2)} C}, \\
I^{(1)}I^{(2)} &= C^{(1)}C^{(2)} \cos A^{(2)}, \\
C^{(2)}C^{(3)} &= \frac{C^{(1)}C^{(2)} \sin C^{(2)} C^{(1)}C^{(3)}}{\sin C^{(2)} C^{(3)} C^{(1)}},
\end{aligned}$$

and generally

$$\begin{aligned}
I^{(i)}I^{(i+1)} &= C^{(i)}C^{(i+1)} \cos A^{(i+1)} \\
C^{(i+1)}C^{(i+2)} &= \frac{C^{(i)}C^{(i+1)} \sin C^{(i+1)} C^{(i)}C^{(i+2)}}{\sin C^{(i+1)} C^{(i+2)} C^{(i)}}.
\end{aligned}$$

Let $\alpha^{(1)}$ and $\beta^{(1)}$ be the errors of the angles opposed to the sides $CC^{(1)}$ and $C^{(1)}C^{(2)}$ in the first triangle. Let $\alpha^{(2)}$ and $\beta^{(2)}$ be the errors of the angles opposed to the sides $C^{(1)}C^{(2)}$ and $C^{(1)}C^{(3)}$ of the second triangle, and thus consecutively. By designating by δ a variation relative to these errors, one will have

$$\begin{aligned}
\frac{\delta I^{(i)}I^{(i+1)}}{I^{(i)}I^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} - \delta A^{(i+1)} \tan A^{(i+1)}, \\
\frac{\delta C^{(i)}C^{(i+1)}}{C^{(i)}C^{(i+1)}} &= \frac{\delta C^{(i)}C^{(i-1)}}{C^{(i)}C^{(i-1)}} + \beta^{(i)} \cot C^{(i+1)}C^{(i-1)}C^{(i)} \\
&\quad - \alpha^{(i)} \cot C^{(i)}C^{(i+1)}C^{(i-1)}.
\end{aligned}$$

One has further, by supposing the angles $A^{(i)}$ relative to the acute angles that the sides of the triangles form with the line $AA^{(1)}$, \dots ,

$$\delta A^{(i+1)} + \delta A^{(i)} + \delta C^{(i-1)}C^{(i)}C^{(i+1)} = 0;$$

we will suppose here that the errors $\alpha^{(i)}$ and $\beta^{(i)}$ of the angles $C^{(i+1)}C^{(i-1)}C^{(i)}$, $C^{(i)}C^{(i+1)}C^{(i-1)}$ of the triangle $C^{(i-1)}C^{(i)}C^{(i+1)}$ are those which remain, when one has subtracted from each angle of the triangle the third of the sum of the errors of the three angles. Then one has

$$\delta C^{(i-1)}C^{(i)}C^{(i+1)} = -\alpha^{(i)} - \beta^{(i)},$$

this which gives

$$\delta A^{(i+1)} = -\delta A^{(i)} + \alpha^{(i)} + \beta^{(i)};$$

one will have therefore

$$\begin{aligned} \delta A^{(i+1)} &= \alpha^{(i)} - \alpha^{(i-1)} + \alpha^{(i-2)} - \dots \mp \alpha^{(1)} \\ &+ \beta^{(i)} - \beta^{(i-1)} + \beta^{(i-2)} - \dots \pm \delta A^{(1)}, \end{aligned}$$

the superior sign having place if i is even, and the inferior if i is odd.

One will have next, by observing that

$$\cot C^{(i)} C^{(i-1)} C^{(i+1)} = \cot C^{(i)} C^{(i+1)} C^{(i-1)} = \cot A^{(i)}$$

and that $A^{(i)} = A^{(1)}$.

$$\frac{\delta C^{(i)} C^{(i+1)}}{C^{(i)} C^{(i+1)}} = \frac{\delta C C^{(1)}}{C C^{(1)}} + (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta^{(1)} - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha^{(1)}) \cot A^{(1)};$$

one will have therefore

$$\begin{aligned} \frac{\delta I^{(i)} I^{(i+1)}}{I^{(i)} I^{(i+1)}} &= \frac{\delta C C^{(1)}}{C C^{(1)}} + (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta^{(1)} - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha^{(1)}) \cot A^{(1)} \\ &- (\alpha^{(i)} - \alpha^{(i-1)} + \dots \mp \alpha^{(1)} + \beta^{(i)} - \beta^{(i-1)} + \dots \mp \beta^{(1)} \pm \delta A^{(i)}) \tan A^{(1)}. \end{aligned}$$

We suppose now that one has measured a base AC situated in a manner that the angle $CAC^{(1)}$ is equal to the angle $CA^{(1)}A$. The first of these angles determine the position of the line $AA^{(1)}$ with respect to the base, and it is supposed known. By naming α and β the errors of the angles $CC^{(1)}A$ and $CAC^{(1)}$, one will have

$$\begin{aligned} \delta A^{(1)} &= \alpha + \beta, \\ \frac{\delta C C^{(1)}}{C C^{(1)}} &= \beta \cot CAC^{(1)} - \alpha \cot CC^{(1)}A. \end{aligned}$$

We make

$$\begin{aligned} \cot CAC^{(1)} &= \cot A + h, \\ \cot CC^{(1)}A &= \cot A + h'; \end{aligned}$$

we will have, by designating by b the base AC and by a the straight line $II^{(i)}$,

$$h = \frac{b}{2a \sin A} - \frac{1}{\sin 2A},$$

$$h' = \frac{a}{2b \sin A \cos^2 A} - \frac{1}{\sin 2A}.$$

we will have next

$$\delta A^{(i+1)} = \alpha^{(i)} - \alpha^{(i-1)} + \dots \pm \alpha + \beta^{(i)} - \beta^{(i-1)} + \dots \pm \beta,$$

$$\frac{\delta I^{(i)} I^{(i+1)}}{I^{(i)} I^{(i+1)}} = (\beta^{(i)} + \beta^{(i-1)} + \dots + \beta - \alpha^{(i)} - \alpha^{(i-1)} - \dots - \alpha) \cot A$$

$$- (\alpha^{(i)} - \alpha^{(i-1)} + \dots \pm \alpha + \beta^{(i)} - \beta^{(i-1)} + \dots \pm \beta) \tan A + h\beta - h'\alpha.$$

The variation of the total length $II^{(i+1)}$ will be therefore

$$\delta II^{(i+1)} = [(i+1)(\beta - \alpha) + i(\beta^{(i)} - \alpha^{(i)} + \dots + (\beta^{(i)} - \alpha^{(i)}))] a \cot A$$

$$+ (i+1) h a \beta - (i+1) h' a \alpha$$

$$- (\alpha^{(i)} + \alpha^{(i-2)} + \alpha^{(i-4)} + \dots + \beta^{(i)} + \beta^{(i-2)} + \beta^{(i-4)} + \dots) a \tan A.$$

The quantity

$$p^2 - pq + q^2 + p^{(1)2} - p^{(1)}q^{(1)} + q^{(1)2} + \dots + p^{(i)2} - p^{(i)}q^{(i)} + q^{(i)2}$$

becomes thus, by neglecting the terms of order i ,

$$\frac{(i+1)(i+2)(2i+3)}{2} a^2 \cot^2 A + 3(h+h')(i+1)^2 a^2 \cot A$$

$$+ (h^2 + hh' + h'^2)(i+1)^2 a^2.$$

We name Q this quantity; the probability that the error of the line $II^{(i+1)}$ is contained within the limits $\pm s$ will be, by that which precedes,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3s}{2\theta} \sqrt{\frac{i+1}{Q}},$$

θ^2 being the sum of the squares of the errors of the sum of the three angles of the $i + 1$ triangles.

We suppose that one has, as for the part of the meridian of which we have spoken previously, twenty-six triangles, this which gives $i = 25$. We suppose further that the length $II^{(i+1)}$ is that of this part of the meridian or of 466006^m; then one will have

$$a = \frac{466006}{26}.$$

By taking for unity the base measured near to Perpignan, which is of 11706^m, 40 right-angled the isosceles triangles $CC^{(1)}C^{(2)}$, $C^{(1)}C^{(2)}C^{(3)}$, ... this which gives $\tan A = \cot A = 1$, one finds

$$Q = 48207, 6.$$

One has seen previously that the twenty-six triangles which join the base of Perpignan to Formentera give

$$Q = 48350, 6;$$

these two values of Q are very little different, and as the equally probable errors are proportionals to the square roots of these values, one sees that one is able to wager one against one that the errors of the entire measure are contained within the limits $\pm 8^m, 1$. Under this relation, the case that we examine represents perfectly the measure of the arc of the meridian from the base of Perpignan to Formentera.

3. We suppose now that one measures, toward the last extremity of the line $II^{(i+1)}$, a base $C^{(i+1)}A^{(i+2)}$ equal to the base CA , and put in a manner that the angle $C^{(i+1)}C^{(i)}A^{(i+2)}$ is equal to the angle $CC^{(1)}A$, and that the angle $C^{(i)}A^{(i+2)}C^{(i+1)}$ is equal to the angle $CAC^{(1)}$. In designating by $\alpha^{(i+1)}$ and $\beta^{(i+1)}$ the errors of the angles $C^{(i+1)}C^{(i)}A^{(i+2)}$ and $C^{(i)}A^{(i+2)}C^{(i+1)}$, the equation

$$C^{(i+1)}A^{(i+2)} = C^{(i+1)}C^{(i)} \frac{\sin C^{(i+1)}C^{(i)}A^{(i+2)}}{\sin C^{(i)}A^{(i+2)}C^{(i+1)}}$$

will give

$$\frac{\delta C^{(i+1)} A^{(i+2)}}{C^{(i+1)} A^{(i+2)}} = \frac{\delta C^{(i)} C^{(i+1)}}{C^{(i)} C^{(i+1)}} + \alpha^{(i+1)} \cot C C^{(1)} A - \beta^{(i+1)} \cot C A C^{(1)},$$

this which gives

$$\begin{aligned} \frac{\delta C^{(i+1)} A^{(i+2)}}{C^{(i+1)} A^{(i+2)}} &= (\beta^{(1)} + \beta^{(2)} + \dots + \beta^{(i)} - \alpha^{(1)} - \alpha^{(2)} - \dots - \alpha^{(i)} \cot A \\ &\quad + \beta(h + \cot A) - \alpha(h' + \cot A) \\ &\quad + \alpha^{(i+1)}(h' + \cot A) - \beta^{(i+1)}(h + \cot A). \end{aligned}$$

That which we have designated in n° 2 of the second Supplement by $l, l^{(1)}, \dots, m, m^{(1)}, \dots$ becomes

$$\begin{aligned} l &= -(1 + h')b, & m &= (1 + h)b, \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ \dots\dots\dots, & & \dots\dots\dots, & \\ l^{(i)} &= -b, & m^{(i)} &= b, \\ l^{(i+1)} &= (1 + h')b, & m^{(i+1)} &= -(1 + h)b; \end{aligned}$$

the quantity that we have designated by $Sf^{(i)2}$ in the section cited or by

$$l^2 - ml + m^2 + l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2} + \dots$$

becomes here

$$3(i + 2)b^2 + 6(h + h')b^2 + 2(h^2 + hh' + h'^2)b^2.$$

The quantity that we have named $Sr^{(i)} f^{(i)}$ in the same section, or

$$l(p - \frac{1}{2}q) + m(q - \frac{1}{2}p) + l^{(1)}(p^{(1)} - \frac{1}{2}q^{(1)}) + m^{(1)}(q^{(1)} - \frac{1}{2}p^{(1)}) + \dots,$$

becomes, by neglecting the terms which do not have i for coefficient,

$$\frac{3(i + 1)(i + 2)}{2} ab + 3(i + 1)(h + h')ab + (i + 1)(h^2 + hh' + h'^2)ab;$$

by representing therefore, as above, by λ the excess of the measured base $C^{(i+1)} A^{(i+2)}$ on the calculated base, and by s the excess of the true length of the line $II^{(i+1)}$ over that calculated length, one will have

$$s = \frac{\lambda S r^{(i)} f^{(i)}}{S f^{(i)2}} = \frac{(i+1)a\lambda}{2b};$$

it is necessary, consequently, to add to the calculated length of the line $II^{(i+1)}$ the product of λ by the ratio of the half of this line to the base b ; this which returns to calculating the first half of the line $II^{(i+1)}$ with the base AC , and the second half with the base $A^{(i+2)}C^{(i+1)}$. This process will be generally exact, whatever was the magnitude and the disposition of the triangles which unite the two bases, if the parts of $Sr^{(i)}f^{(i)}$ and of $Sf^{(i)2}$ corresponding to these halves were respectively equals. It is this process that we adopted in the Commission which fixed the length of the meter; and, in ignorance where we were then of the true theory of these corrections, it was most convenient; but it did not make known the correction of the diverse parts of the total arc $II^{(i+1)}$. For this, it is necessary to correct the angles of each triangle, or to determine the corrections $\alpha, \beta, \alpha^{(1)}, \beta^{(1)}, \dots$ which result from the excess λ of the second base observed over this base calculated after the first. I have given, in the second Supplement, these corrections, by supposing the law of errors of the observations of the angles proportional to the exponential $e^{-k(\alpha+\frac{1}{3}T)^2}$, k being a constant, T being the sum of the errors of the three angles of the triangle, $\alpha + \frac{1}{3}T$, $\beta + \frac{1}{3}T$ and $\frac{1}{3}T - \alpha - \beta$ being the errors of each of the angles. One has seen, in the Supplement cited, that the supposition of this law of probability must be admitted when the angles have been measured with the repeating circle, and that then one has

$$\alpha^{(s)} = \frac{l^{(s)} - \frac{1}{2}m^{(s)}}{F} \lambda, \quad \beta^{(s)} = \frac{m^{(s)} - \frac{1}{2}l^{(s)}}{F} \lambda,$$

by designating by F the sum of all the quantities $l^2 - ml + m^2$, $l^{(1)2} - m^{(1)}l^{(1)} + m^{(1)2}$, \dots I will demonstrate here that these corrections have place, whatever be the law of probability of the errors.

For this, I designate this law by $\phi(\alpha+\frac{1}{3}T)^2$: by supposing it the same for the positive errors and for the negative errors, its expression must contain only some even powers of these errors. The law of probability of the simultaneous values of α and β will be thus proportional to the product

$$\phi(\alpha+\frac{1}{3}T)^2 \phi(\beta+\frac{1}{3}T)^2 \phi(\frac{1}{3}T - \alpha - \beta)^2.$$

If one develops this product, with respect to the powers of α and of β , by arresting oneself at the squares and at the products of these quantities, one will have

$$[\phi(\frac{1}{9}T)]^3 + (\alpha^2 + \alpha\beta + \beta^2)\phi(\frac{1}{9}T^2) \\ \times \left\{ 2\phi(\frac{1}{9}T^2)\phi'(\frac{1}{9}T^2) - \frac{4}{9}T^2[\phi'(\frac{1}{9}T^2)]^2 + \frac{4}{9}T^2\phi(\frac{1}{9}T^2)\phi''(\frac{1}{9}T^2) \right\}$$

$\phi'(x)$ expressing $\frac{d\phi(x)}{dx}$, and $\phi''(x)$ expressing $\frac{d^2\phi(x)}{dx^2}$. T being able to be supposed to vary from $-\infty$ to $T = \infty$, one will multiply the preceding function by dT and one will integrate within these limits; one will have thus for the probability of the simultaneous values of α and β a quantity of the form

$$H - H'(\alpha^2 + \alpha\beta + \beta^2).$$

This probability will be therefore proportional to

$$1 - \frac{H'}{H}(\alpha^2 + \alpha\beta + \beta^2).$$

The probability of the simultaneous existence of $\alpha, \beta, \alpha^{(1)}, \beta^{(1)}, \dots$ will be proportional to the product of the quantities

$$1 - \frac{H'}{H}(\alpha^2 + \alpha\beta + \beta^2), \\ 1 - \frac{H'}{H}(\alpha^{(1)2} + \alpha^{(1)}\beta^{(1)} + \beta^{(1)2}), \\ \dots\dots\dots$$

The logarithm of this product is, s being an indeterminate number,

$$- \frac{H'}{H}S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2}) - \dots;$$

this product is at its maximum if the preceding term is at its minimum, or if the function

$$S(\alpha^{(s)2} + \alpha^{(s)}\beta^{(s)} + \beta^{(s)2})$$

is the smallest possible, the quantities $\alpha, \beta, \alpha^{(1)}, \dots$ satisfying besides the equation

$$\lambda = l\alpha + m\beta + l^{(1)}\alpha^{(1)} + m^{(1)}\beta^{(1)} + \dots$$

One is able to give to this function the form

$$\frac{1}{4}S \left\{ \left(2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F} \right)^2 + \frac{3}{4} \left[\alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F} \right]^2 \right\} + \frac{3}{4} \frac{\lambda^2}{F};$$

this function is evidently at its minimum if one supposes

$$2\beta^{(s)} + \alpha^{(s)} - \frac{3m^{(s)}\lambda}{2F} = 0, \quad \alpha^{(s)} - \frac{(l^{(s)} - \frac{1}{2}m^{(s)})\lambda}{F} = 0;$$

whence one draws generally

$$\alpha^{(s)} = (l^{(s)} - \frac{1}{2}m^{(s)}) \frac{\lambda}{F}, \quad \beta^{(s)} = (m^{(s)} - \frac{1}{2}l^{(s)}) \frac{\lambda}{F}.$$

In the case that we just considered, one has

$$\begin{aligned} \alpha &= -\frac{\lambda b}{F} \left(\frac{3}{2} + h' + \frac{1}{2}h \right), & \beta &= \frac{\lambda b}{F} \left(\frac{3}{2} + h + \frac{1}{2}h' \right), \\ \alpha^{(1)} &= \alpha^{(2)} = \dots = \alpha^{(i)} = -\frac{\frac{3}{2}b\lambda}{F}, & \beta^{(1)} &= \beta^{(2)} = \dots = \beta^{(i)} = \frac{\frac{3}{2}b\lambda}{F}, \\ \alpha^{(i+1)} &= \frac{\lambda b}{F} \left(\frac{3}{2} + h' + \frac{1}{2}h \right), & \beta^{(i+1)} &= -\frac{\lambda b}{F} \left(\frac{3}{2} + h + \frac{1}{2}h' \right); \end{aligned}$$

thus by these corrections all the triangles other than those which have one of the bases for one of their sides will remain right-angled.

The probability of the error $\pm u$ of the line $II^{(i+1)}$, corrected by the second base, will be, by the section cited in the second Supplement,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Q \frac{(Sr^{(i)}f^{(i)})^2}{Sf^{(i)^2}}}},$$

which becomes here

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Q'}},$$

by designating by Q' the function

$$\frac{(i+1)(i+2)(i+3)}{4}a^2 + \frac{3}{2}(i+1)^2(h+h')a^2 + \frac{1}{2}(i+1)^2(h^2+hh'+h'^2)a^2.$$

The equally probable errors being proportionals to the square roots of Q and of Q' , one sees that they are diminished and nearly reduced to half of the measure of a second base.

The probability of an error $\pm \lambda$ in the measure of a second base is, by the second Supplement,

$$\frac{2 \int dt e^{-t^2}}{\sqrt{\pi}},$$

the integral being taken from t null to

$$t = \frac{3u}{2\theta} \sqrt{\frac{i+1}{Sf^{(i)^2}}};$$

and $f^{(i)^2}$ is equal to

$$3(i+1)b^2 + 6(h+h')b^2 + 2(h^2+hh'+h'^2)b^2.$$

In the present case where $i = 25$, this quantity becomes

$$86,8030b^2;$$

the equally probable errors in the measures of the arc $II^{(i+1)}$ and of a new base equal to the first are therefore in the ratio of \sqrt{Q} to $\sqrt{86,8030}$; whence it follows that there are odds one against one that the error of a new base will be contained within the limits $\pm 0^m, 34236$, or to very nearly $\pm \frac{1}{3}^m$. These are the same limits which result from the angles of the twenty-six triangles which reunite the base of Perpignan to Formentera. Thus, under this relation again, the hypothetical case, which we have just examined, accords with that which this chain of triangles gives.

4. I will consider the zenithal distances of the vertices of the triangles and the leveling which results from it. From one same vertex such as $C^{(2)}$, one is able to observe the four points C , $C^{(1)}$, $C^{(3)}$, $C^{(4)}$. We name f the distance $CC^{(1)}$ and h the base $CC^{(2)}$ of the isosceles triangle; all the triangles being supposed equal, if one names $x^{(i)}$ the height of $C^{(i)}$ above of the level of the sea, the observed distance of $C^{(i-2)}$ at the zenith of $C^{(i)}$ being designated by θ , the true distance will be quite nearly, the triangles being able to be supposed horizontal,

$$\theta + \frac{hu}{R} + \frac{h\epsilon}{R},$$

u being the factor by which one must multiply the angle $\frac{h}{R}$ in order to have the terrestrial refraction at the point $C^{(i)}$, R being the radius of the Earth and ϵ being the error of u . I take account here only of this error, as being much greater than that of θ . If one names similarly θ' the zenithal distance of $C^{(i)}$, observed from $C^{(i-2)}$, the true distance will be

$$\theta' + \frac{hu}{R} + \frac{h\epsilon'}{R},$$

ϵ' being the error of u in this observation. One will have

$$\theta + \theta' + \frac{2hu}{R} + \frac{h}{R}(\epsilon + \epsilon') = \pi + \frac{h}{R};$$

one will have next

$$x^{(i)} - x^{(i-2)} = \frac{h}{2}(\theta - \theta') + \frac{h^2}{2R}(\epsilon - \epsilon').$$

If one names similarly θ'' the zenithal distance of $C^{(i-1)}$ observed from $C^{(i)}$, the true distance will be

$$\theta'' + \frac{fu}{R} + \frac{f\epsilon''}{R},$$

ϵ'' being the error of u in this observation. By naming further θ''' and ϵ''' the same quantities relative to the zenithal distance of $C^{(i)}$, observed from $C^{(i-1)}$, one will have

$$\theta'' + \theta''' + \frac{2fu}{R} + \frac{f}{R}(\epsilon'' + \epsilon''') = \pi + \frac{f}{R},$$

$$x^{(i)} - x^{(i-1)} = \frac{f}{2}(\theta'' - \theta''') + \frac{f^2}{2R}(\epsilon'' - \epsilon''').$$

As I myself propose here only to examine what degree of confidence one must accord to this kind of leveling, I will make $h = f$, this which returns to supposing all the triangles equilateral. I will take, moreover, $\frac{h^2}{2R}$ for unity of distance: by making next $\epsilon - \epsilon' = \lambda^{(i)}$, $\epsilon'' - \epsilon''' = \gamma^{(i)}$, one will have two equations of the form

$$(A) \quad \begin{cases} x^{(i)} - x^{(i-1)} = \gamma^{(i)} + p^{(i)}, \\ x^{(i)} - x^{(i-2)} = \lambda^{(i)} + q^{(i)}. \end{cases}$$

The first of these equations extend from $i = 1$ to $i = n + 1$, n being the number of triangles. The second equation extends from $i = 2$ to $i = n + 1$. It is necessary now to conclude from this system of equations the most advantageous value of $x^{(n+1)} - x^{(0)}$, the elevation $x^{(0)}$ of the point C above the sea being supposed known. For this, one will multiply the first of the equations (A) by $f^{(i)}$ and the second by $g^{(i)}$, $f^{(i)}$ and $g^{(i)}$ being indeterminate constants. In the system of these equations added all together, the coefficient of $x^{(i)}$ will be $f^{(i)} - f^{(i+1)} + g^{(i)} - g^{(i+2)}$. By equating it to zero and observing that $g^{(i+2)} - g^{(i)} = \Delta g^{(i+1)} + \Delta g^{(i)}$, Δ being the characteristic of finite differences, one will have, by integrating,

$$f^{(i)} = a - g^{(i)} - g^{(i+1)},$$

a being a constant. But, the values of $g^{(i)}$ beginning to have place only when $i = 2$, this expression of $f^{(i)}$ is able to serve only when $i = 2$. In order to have the value of $f^{(1)}$, one will observe that the equality to zero of the coefficient of $x^{(1)}$ gives

$$f^{(1)} = f^{(2)} + g^{(3)};$$

substituting, instead of $f^{(2)}$, $a - g^{(2)} - g^{(3)}$, one will have

$$f^{(1)} = a - g^{(2)}.$$

Next, the preceding expression of $f^{(i)}$ extends only to $i = n$; but, relatively to

$i = n + 1$, one must observe that the coefficient of $x^{(n+1)}$ must be unity, this which gives

$$f^{(n+1)} + g^{(n+1)} = 1$$

or

$$f^{(n+1)} = 1 - g^{(n+1)};$$

the equality to zero of the coefficient of $x^{(n)}$ gives $f^{(n)} = f^{(n+1)} - g^{(n)}$, or $f^{(n)} = 1 - g^{(n)} - g^{(n+1)}$. By comparing this expression to the one here $f^{(n)} = a - g^{(n)} - g^{(n+1)}$, one will have $a = 1$. The error of the value of $x^{(n+1)}$ will be thus

$$\begin{aligned} & f^{(1)}\gamma^{(1)} + f^{(2)}\gamma^{(2)} + \dots + f^{(n+1)}\gamma^{(n+1)} \\ & + g^{(2)}\lambda^{(2)} + g^{(3)}\lambda^{(3)} + \dots + g^{(n+1)}\lambda^{(n+1)} \end{aligned}$$

The values of $\gamma^{(1)}, \gamma^{(2)}, \dots, \lambda^{(1)}, \lambda^{(2)}, \dots$ being evidently subject to the same law of probability, if one names s this error and if one makes

$$\begin{aligned} H = & f^{(1)2} + f^{(2)2} + \dots + f^{(n+1)2} \\ & + g^{(1)2} + g^{(3)2} + \dots + g^{(n+1)2}, \end{aligned}$$

the probability of the error s will be proportional, by n° 20 of Book II, to an exponential of the form

$$c^{-\frac{Ks^2}{H}},$$

K being a constant dependent on the law of probability of $\gamma^{(i)}$ and $\lambda^{(i)}$.

It is necessary to determine the constants of H , in a manner that H is a minimum. Now one has

$$\begin{aligned} H = & (1 - g^{(2)})^2 + (1 - g^{(2)} - g^{(3)})^2 + \dots + (1 - g^{(n)} - g^{(n+1)})^2 \\ & + (1 - g^{(n+1)})^2 + g^{(2)2} + g^{(3)2} + \dots + g^{(n+1)2}; \end{aligned}$$

by equating to zero the coefficient of the differential of $g^{(i)}$, one has

$$(1) \quad g^{(i+1)} + 3g^{(i)} + g^{(i-1)} = 2.$$

This equation has place from $i = 3$ to $i = n$. The equality to zero of the coefficient of $dg^{(2)}$ gives

$$g^{(3)} + 3g^{(2)} = 2,$$

and the equating to zero of the coefficient of $dg^{(n+1)}$ gives

$$3g^{(n+1)} + g^{(n)} = 2,$$

this which returns to consider the general equation (1) as having place from $i = 2$ to $i = n + 1$, and to suppose null $g^{(1)}$ and $g^{(n+2)}$. The integration of equation (1) in the finite differences gives

$$g^{(i)} = \frac{2}{5} + Al^{i-1} + A'l^{i-1},$$

l and l' being the two roots $-\frac{3}{2} - \frac{1}{2}\sqrt{5}$, $-\frac{3}{2} + \frac{1}{2}\sqrt{5}$ of the equation

$$y^3 + 3y + 1 = 0;$$

A and A' are two arbitraries such that $g^{(i)}$ becomes null when $i = 1$ and when $i = n + 2$. One has therefore

$$\begin{aligned} Al^{n+1} + A'l^{n+1} &= -\frac{2}{5}, \\ A + A' &= -\frac{2}{5}. \end{aligned}$$

l^{n+1} is an extremely great quantity when n is a great number and, l'^{n+1} being $\frac{1}{l^{n+1}}$, one sees that A is then an excessively small quantity and that thus $A' = -\frac{2}{5}$. One has next

$$f^{(i)} = \frac{1}{5} - Al^{i-1}(1+l) - A'l^{i-1}(1+l').$$

Thence it is easy to conclude that one has, very nearly and without fear $\frac{1}{25}$ of error.

$$H = \frac{n+1}{5},$$

and that thus the exponential proportional to the probability of error s is

$$c^{-\frac{5Ks^2}{n+1}};$$

one is able therefore thus to determine this probability.

One has concluded the value of $x^{(n+1)}$ of the system of equations (A) by the following process.

The system of equations (A) gives

$$x^{(1)} - x^{(0)} = p^{(1)} + \gamma^{(1)};$$

whence one draws

$$x^{(1)} = p^{(1)} + x^{(0)} + \gamma^{(1)}.$$

One has next the two equations

$$\begin{aligned} x^{(2)} - x^{(1)} &= p^{(2)} + \gamma^{(2)}, \\ x^{(2)} - x^{(0)} &= q^{(2)} + \lambda^{(2)}; \end{aligned}$$

this which gives

$$x^{(2)} = \frac{1}{2}x^{(1)} + \frac{1}{2}x^{(0)} + \frac{1}{2}(p^{(2)} + q^{(2)}) + \frac{1}{2}\gamma^{(2)} + \frac{1}{2}\lambda^{(2)}.$$

One has the two equations

$$\begin{aligned} x^{(3)} - x^{(2)} &= p^{(3)} + \gamma^{(3)}, \\ x^{(3)} - x^{(1)} &= q^{(3)} + \lambda^{(3)}; \end{aligned}$$

this which gives

$$x^{(3)} = \frac{1}{2}x^{(2)} + \frac{1}{2}x^{(1)} + \frac{1}{2}(p^{(3)} + q^{(3)}) + \frac{1}{2}\gamma^{(3)} + \frac{1}{2}\lambda^{(3)}.$$

By continuing thus, one will have $x^{(n+1)}$. The quantities $\gamma^{(m)}$ and $\lambda^{(m)}$ commence to be introduced into this expression only with the two values of $x^{(m)} - x^{(m-1)}$ and of $x^{(m)} - x^{(m-2)}$. We designate by $k^{(r)}$ the coefficient of $\gamma^{(m)}$ in the expression of $x^{(m+r)}$; this expression is

$$x^{(m+r)} = \frac{1}{2}x^{(m+r-1)} + \frac{1}{2}x^{(m+r-2)} + \frac{1}{2}(p^{(m+r)} + q^{(m+r)}) + \frac{1}{2}\gamma^{(m+r)} + \frac{1}{2}\lambda^{(m+r)};$$

by substituting for $x^{(m+r)}$, $x^{(m+r-1)}$, $x^{(m+r-2)}$ the parts of their values relative to $\gamma^{(m)}$, the comparison of the coefficients of this quantity will give

$$k^{(r)} = \frac{1}{2}k^{(r-1)} + \frac{1}{2}k^{(r-2)};$$

whence one draws, by integrating,

$$k^{(r)} = A + A' \left(-\frac{1}{2} \right)^{r-1},$$

A and A' being two arbitraries. In order to determine them, we will observe that, r being null, one has $k^{(0)} = \frac{1}{2}$, and that, r being 1, one has

$$k^{(1)} = \frac{1}{2}k^{(0)} = \frac{1}{4};$$

thence one draws

$$A = \frac{1}{3}, \quad A' = -\frac{1}{12};$$

thus, in the value of $x^{(n+1)}$, where $r = n + 1 - m$, one will have, for the coefficient $k^{(n+1-m)}$ of $\gamma^{(m)}$,

$$k^{(n+1-m)} = \frac{1}{3} - \frac{1}{12} \left(-\frac{1}{2} \right)^{n-m};$$

the coefficient of $\lambda^{(m)}$ in the same value will be evidently the same. Thus the expression of $x^{(n+1)}$ will be a known quantity, plus the sequence

$$k^{(n)}\gamma^{(1)} + k^{(n-1)}(\gamma^{(2)} + \lambda^{(2)}) + \dots + k^{(0)}(\gamma^{(n+1)} + \lambda^{(n+1)}).$$

We designate by s this error and by H the sum of the squares of the coefficients of $\gamma^{(1)}, \gamma^{(2)}, \dots, \lambda^{(2)}, \lambda^{(3)}, \dots$; the probability of s will be proportional to $c^{-\frac{Ks^2}{H}}$. One has, very nearly,

$$H = \frac{2}{9}(n + 1);$$

thus the probability of s is very nearly proportional to $c^{\frac{-9Ks^2}{2(n+1)}}$; the equally probable errors are therefore greater in this process than following the most advantageous method, and nearly in the ratio of $\sqrt{5}$ to $\sqrt{\frac{9}{2}}$; this process approaches therefore much the exactitude of the most advantageous method, and, as the calculation of it is quite simple, we will determine the probability of

the errors to which it exhibits, in the general case where the diverse triangles are neither equal nor equilateral.

If one represents by $m^{(i)}$ the square of $C^{(i-1)}C^{(i)}$ divided by $2R$, and by $n^{(i)}$ the square of $C^{(i-2)}C^{(i)}$ divided similarly by $2R$, the system of equations (A) will be changed into the following:

$$(A') \quad \begin{cases} x^{(i)} - x^{(i-1)} = p^{(i)} + m^{(i)}\gamma^{(i)}, \\ x^{(i)} - x^{(i-2)} = q^{(i)} + n^{(i)}\lambda^{(i)}. \end{cases}$$

The process that we have just examined gives, by following the preceding analysis, the coefficient of $\gamma^{(i)}$ in the expression of $x^{(n+1)}$ equal to

$$\frac{1}{3}m^{(i)} - \frac{1}{12}m^{(i)}\left(-\frac{1}{2}\right)^{n-i}.$$

Similarly, the coefficient of $\lambda^{(i)}$, in the same expression, is

$$\frac{1}{3}n^{(i)} - \frac{1}{12}n^{(i)}\left(-\frac{1}{2}\right)^{n-i};$$

thence it follows that the value of H is, very nearly,

$$\frac{1}{9}S(m^{(i)2} + n^{(i)2}),$$

the integral sign S extending to all the values of i to $i = n + 1$; the probability of an error s , in the expression of $x^{(n+1)}$, is therefore proportional to

$$\frac{e^{-9Ks^2}}{e^{S(m^{(i)2} + n^{(i)2})}}.$$

If one applies to the equations (A') the analysis that we have given above for the case of the most advantageous method, one will find, by multiplying them respectively by $f^{(i)}$ and $g^{(i)}$, the following equation

$$f^{(i)} = 1 - g^{(i)} - g^{(i+1)},$$

and this equation will hold from $i = 1$ to $i = n + 1$, by supposing $g^{(i)}$ and $g^{(i+2)}$ nulls. One will have next the general equation

$$m^{(i)2}g^{(i+1)} + (n^{(i)2} + m^{(i)2} + n^{(i-1)2})g^{(i)} + m^{(i-1)2}g^{(i-1)} = m^{(i)2} + m^{(i-1)2}.$$

This equation holds from $i = 2$ to $i = n + 1$. By combining it with the equations $g^{(1)} = 0$, $g^{(n+2)} = 0$, one will have the values of $f^{(1)}$, $f^{(2)}$, \dots , $f^{(n+1)}$; $g^{(1)}$, $g^{(2)}$, \dots , $g^{(n+2)}$; one will have next

$$H = S(f^{(i)2}m^{(i)2} + g^{(i)2}n^{(i)2}),$$

the sign S comprehending all the values of $f^{(i)}m^{(i)2}$ and of $g^{(i)2}n^{(i)2}$; the probability of an error s in the value of $x^{(n+1)}$ will be proportional to

$$c^{-\frac{Ks^2}{H}},$$

5. It is necessary now to determine the value of K . For this, we will observe that the factor u is determined, by that which precedes, by means of the equation

$$u = \frac{\pi - \theta - \theta' + \frac{h}{R}}{\frac{2h}{R}},$$

and that the error of this expression is $\frac{\epsilon + \epsilon'}{2}$. Each double station furnishes a value of u , and the mean of these values is the value that it is necessary to adopt. If one names i the number of these values, the error to fear will be $S\frac{\epsilon + \epsilon'}{2i}$, the sign S having reference to i quantities $\frac{\epsilon + \epsilon'}{2i}$ related to each double station. Let s be the sum $S\frac{\epsilon + \epsilon'}{2i}$; the probability of s will be, by n° 20 of Book II, proportional to an exponential of the form

$$c^{-\frac{K's^2}{i}},$$

and, if one names q the sum of the squares of the differences of each partial value to its mean value, one will have

$$K' = \frac{i}{2q}.$$

One has, by that which precedes, the probability of the error of a value s' of the function $S\frac{\epsilon - \epsilon'}{2i}$ proportional to the exponential

$$c^{-\frac{4Ks'^2}{i}},$$

the sign S extending to i quantities of the form $\frac{\epsilon - \epsilon'}{2}$. Now, the errors ϵ and $-\epsilon$ being supposed equally probable, it is clear that the same values of $S\frac{\epsilon + \epsilon'}{2i}$ and of

$S \frac{\epsilon - \epsilon'}{2i}$ are equally probable; one has therefore

$$4K = K',$$

this which gives

$$K = \frac{i}{8q}.$$

The forty-five first values of u , given in the second Volume of the *Base du Système métrique* (p. 771), and which are founded on some observations made in the month of the year where one observes most often, give, for its mean value,

$$u = 0,07818,$$

and the sum q of the squares of the differences of these values to the mean is 0,04900629; i being here equal to 45, one has

$$K = \frac{45}{0,39205032} = 114,781.$$

If one supposes the number n of triangles equal to 25 and if one makes all the sides equal to 20000^m, one will have 240000^m for the distance from $x^{(26)}$ to $x^{(0)}$; that is nearly the distance from Paris to Dunkerque. In this case, the quantity $\frac{f^2}{2R}$, taken for unity of distance, is 31^m, 416. Thence one concludes that the odds are one against one that the error on the height $x^{(26)}$ is contained within the limits $\pm 3^m, 1839$. There are odds nine against one that it is contained within the limits $\pm 7^m, 761$; one is not able therefore then to respond with a sufficient probability that this error will not exceed $\pm 8^m$.

The chain of triangles that we have just considered is much more favorable to the determination of the height of its last point than that of which Delambre has made use, in the Work cited, in order to determine the height of the Pantheon above the level of the sea. By considering this last chain, one sees that one is not able to respond, with a sufficient probability, that the error on this height will not exceed $\pm 16^m$.

6. One sees, by that which precedes, that the great triangles, which are very proper to the measure of terrestrial degrees, are too small in order to determine the respective heights of the diverse stations. Thus, in the case of a chain of equilateral triangles of which f is the length of each side, the equally probable errors of the difference of level of two extreme stations being proportional to

$\frac{f^2\sqrt{n+1}}{2R}$, n being the number of triangles, if one names a the distance of these two stations, one will have, by supposing $n + 1$ even,

$$a = \frac{1}{2}(n + 1)f;$$

$\frac{f^2\sqrt{n+1}}{2R}$ will be therefore proportional to $\frac{1}{(n+1)^{\frac{3}{2}}}$; the equally probable errors will be therefore proportional to this fraction. Thus, by quadrupling the number of triangles, they will become eight times smaller; but then the errors due to the observations of the angles become comparable to the errors due to the variability of the terrestrial refractions. We examine how one is able to have regard to the same time to these two kinds of errors.

We consider a sequence of points $C, C^{(1)}, C^{(2)}, \dots$. Let $h^{(0)}$ be the distance from C to $C^{(1)}$; $h^{(1)}$ the distance from $C^{(1)}$ to $C^{(2)}$; $h^{(2)}$ the distance from $C^{(2)}$ to $C^{(3)}$, and thus consecutively. We imagine that from the point $C^{(i)}$ one observes $C^{(i+1)}$, and reciprocally. The zenithal distance from $C^{(i+1)}$, observed from $C^{(i)}$ will be, by that which precedes,

$$\theta + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon}{R} + \alpha,$$

ϵ being the error of u and α being that of the observed angle θ . The zenithal distance of $C^{(i)}$, observed from $C^{(i+1)}$, will be

$$\theta' + \frac{h^{(i)}u}{R} + \frac{h^{(i)}\epsilon'}{R} + \alpha',$$

ϵ' and α' being the errors of u and of θ' in the observation made at the point $C^{(i+1)}$. One will have therefore the two equations

$$\begin{aligned} \theta + \theta' + \frac{2h^{(i)}u}{R} + \frac{h^{(i)}u}{R}(\epsilon + \epsilon') + \alpha + \alpha' &= \pi + \frac{h^{(i)}}{R}, \\ x^{(i+1)} - x^{(i)} &= \frac{\theta - \theta'}{2}h^{(i)} + \frac{h^{(i)2}}{2R}(\epsilon - \epsilon') + \frac{1}{2}h^{(i)}(\alpha - \alpha'). \end{aligned}$$

We designate as above $\epsilon - \epsilon'$ by $\gamma^{(i)}$, and we make $\alpha - \alpha'$ equal to $\lambda^{(i)}$; one will have, for the elevation $x^{(n+1)} - x^{(0)}$ of the point $C^{(n+1)}$ above C , an expression of the form

$$x^{(n+1)} - x^{(0)} = M + S \frac{h^{(i)2}}{2R} \gamma^{(i)} + S \frac{1}{2} h^{(i)} \lambda^{(i)},$$

the integral sign S having reference to all the values of i , from $i = 0$ to $i = n$. The error of this value of $x^{(n+i)}$ is

$$S \frac{h^{(i)2} \gamma^{(i)}}{2R} + S \frac{h^{(i)}}{2} \lambda^{(i)}.$$

It is necessary now to determine the probability of this error that we will designate by s . Let generally

$$s = Sm^{(i)} \gamma^{(i)} + Sn^{(i)} \lambda^{(i)};$$

the probability of s will be, by the analysis of n° 20 of Book II of the *Théorie analytique des Probabilités*, proportional to

$$\int d\varpi dx dy \phi(x) \psi(y) c^{-s\varpi \sqrt{-1}} \\ \times [\cos(m^{(0)}x + n^{(0)}y)\varpi \cos(m^{(1)}x + n^{(1)}y)\varpi \cos(m^{(2)}x + n^{(2)}y)\varpi \dots];$$

$\phi(x)$ is the law of probability of a value x of $\gamma^{(0)}$; $\psi(y)$ is the law of probability of a value y of $\lambda^{(0)}$. The negative and positive errors are supposed equally probable: the integrals relative to x and y are taken from negative infinity to positive infinity, and the integral relative to $\varpi = -\pi$ to $\varpi = \pi$. By making

$$2 \int dx \phi(x) = k, \quad \int x^2 dx \phi(x) = k'', \\ 2 \int dy \psi(y) = \bar{k}, \quad \int y^2 dy \psi(y) = \bar{k}'' ,$$

the integrals being taken from x and y null to x and y equal to infinity, the analysis of the section cited will give the probability of s proportional to

$$c^{\frac{-s^2}{\frac{4k''}{k} Sm^{(i)2} + \frac{4\bar{k}''}{\bar{k}} Sn^{(i)2}}}$$

It is easy to conclude generally from the same analysis that, if one makes

$$s = Sm^{(i)} \gamma^{(i)} + Sn^{(i)} \lambda^{(i)} + Sr^{(i)} \delta^{(i)} + \dots,$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, ... being of the errors deriving from different sources, the

probability of s is proportional to the exponential

$$C \frac{-s^2}{\frac{4k''}{k} S m^{(i)2} + \frac{4k''}{k} S n^{(i)2} + \frac{4k''}{k} S r^{(i)2} + \dots}$$

by designating by $\pi(x)$ the probability of an error x due to the third source of error, and making

$$2 \int dx \pi(x) = \bar{k}, \quad \int x^2 dx \pi(x) = \bar{k}'',$$

the integrals being taken from x null to x infinity; and thus of the other errors.

In order to determine, in the present question, the constants $\frac{4k''}{k}$ and $\frac{4k''}{k}$, I will suppose first the second null or very small, relatively to the first, as one is able to do in the great triangulations of the meridian. In this case, the probability of an error s will be, by making $m^{(i)} = 1$, proportional to

$$C \frac{-s^2}{\frac{4k''}{k} n},$$

n being the number of intervals which separate the stations. The probability of a value s' of $S \frac{\epsilon - \epsilon'}{2}$ or of $S \frac{\epsilon + \epsilon'}{2}$, this which corresponds to an error $2s'$ in the value of $S(\epsilon - \epsilon')$, will be proportional to

$$C \frac{-4s'^2}{\frac{4k''}{k} n};$$

but, by that which precedes, this probability is proportional to

$$C \frac{-i s'^2}{2q n};$$

one has therefore

$$\frac{2q}{i} = \frac{k''}{k} \quad \text{or} \quad \frac{4k''}{k} = \frac{8q}{i} = \frac{1}{114,781}.$$

If one supposes now $\frac{k''}{k}$ null and $n^{(i)} = 1$, the probability of a value s' of the sum $S \frac{\alpha' - \alpha}{2}$ will be proportional to

$$C \frac{-4s'^2}{\frac{4k''}{k} n},$$

and the probability of a same value s' of $S(\alpha + \alpha' + \alpha'')$ will be proportional to

$$C \frac{\frac{-2s'^2}{12\bar{k}''n}}{\bar{k}}.$$

If one supposes this law of probability the same as for the errors of the sum of the three angles of a spherical triangle, in the geodesic measures, and which, by n° 1 of the second Supplement, is able to be supposed proportional to

$$C \frac{-(i+2)s'^2}{2\theta^2 n},$$

θ^2 being the sum of the squares of the excess observed in the sum of the errors of the three angles in i triangles, one will have

$$\frac{4\bar{k}''}{\bar{k}} = \frac{4\theta^2}{3(i+2)}.$$

One has, by that which one has seen,

$$\frac{i+2}{\theta^2} = \frac{109}{445,217};$$

hence,

$$\frac{4\bar{k}''}{\bar{k}} = \frac{4}{3} \frac{445,217}{109},$$

a quantity that it is necessary to divide by the square of the number of sexagesimal seconds that this radius contains, and then one has

$$\frac{4\bar{k}''}{\bar{k}} = \frac{1,2801}{10^{10}}.$$

We suppose the distances of the consecutive stations equal to 1200 m; one will find that there are odds one against one that the error on the value of $x^{(n+1)}$ is not above $\pm 0^m, 08555$ when $n = 200$. There are odds one thousand against one that the error is not above $\pm 0^m, 413$.

General method of the Calculation of probabilities, when there are many sources of errors.

The consideration of the two sources of independent error which exist in the operations of the leveling has led me to examine the general case of the

observations subject to many sources of errors. Such are astronomical observations. The greater part are made by means of two instruments, the meridian lunette and the circle, of which the errors must not be supposed to have the same law of probability. In the equations of condition that one deduces from these observations, in order to obtain the elements of the celestial movements, these errors are multiplied by some different coefficients for each source of error and for each equation. The most advantageous systems of factors by which it is necessary to multiply these equations, in order to have the final equations which determine the elements, are no longer, as in the case of a unique source of errors, the coefficients of each element in the equations of condition. The facility with which the analysis that I have given in Book II of my *Théorie des Probabilités* is applied to this general case will show the advantages of this analysis.

We suppose first that one has a system of equations of condition represented by this here

$$p^{(i)}y = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots,$$

y being an element of which one seeks the most advantageous value. If one multiplies the preceding equation by a factor $f^{(i)}$, the reunion of all these products will give for y the expression

$$y = \frac{Sa^{(i)}f^{(i)}}{Sp^{(i)}f^{(i)}} + \frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \dots}{Sp^{(i)}f^{(i)}}$$

The error of y will be

$$\frac{Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \dots}{Sp^{(i)}f^{(i)}}$$

By designating by s this error, its probability will be proportional, by the preceding section, to the exponential

$$c \frac{-s^2(Sp^{(i)}f^{(i)})^2}{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4k''}{k}Sn^{(i)2}f^{(i)2} + \dots}$$

It is necessary to determine $f^{(i)}$ in a manner that

$$\frac{\frac{4k''}{k}Sm^{(i)2}f^{(i)2} + \frac{4k''}{k}Sn^{(i)2}f^{(i)2} + \dots}{(Sp^{(i)}f^{(i)})^2}$$

is a minimum, because it is clear that then the same error s becomes less probable

than in each other system of factors. If one names A the numerator of this fraction, and if one makes $f^{(i)}$ to vary by a quantity dq , one will have, through the condition of the minimum, by equating to zero the differential of this fraction,

$$0 = \frac{\frac{k''}{k}m^{(i)2}f^{(i)} + \frac{\bar{k}''}{\bar{k}}n^{(i)2}f^{(i)} + \dots}{A} - \frac{p^{(i)}}{Sp^{(i)}f^{(i)}}$$

this which gives for $f^{(i)}$ an expression of this form

$$f^{(i)} = \frac{\mu p^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \dots}$$

One is able to make here $\mu = 1$, because, this quantity being independent of i , it affects equally all the multipliers $f^{(i)}$; thus the quantity $f^{(i)}$, by which one must multiply each equation of condition in order to have the most advantageous result, is

$$\frac{p^{(i)}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \dots}$$

and the probability of an error s of this result is proportional to the exponential

$$c \frac{e^{-\frac{s^2}{4}} S \frac{p^{(i)2}}{\frac{k''}{k}m^{(i)2} + \frac{\bar{k}''}{\bar{k}}n^{(i)2} + \dots}}{c}$$

One will have, by the same analysis and by n^o 22 of Book II, the factors by which one must multiply the equations of condition in order to have the most advantageous results, whatever be the number of elements to determine and the number of kinds of errors; one will have similarly the laws of probability of the errors of these results.

We suppose that one has, between two elements x and y , the equation of condition

$$l^{(i)}x + p^{(i)}y = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + r^{(i)}\delta^{(i)} + \dots,$$

$\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, ... being some errors of which the sources are different. By multiplying first this equation by a system $f^{(i)}$ of factors, the reunion of these products will give the final equation

$$xSl^{(i)}f^{(i)} + ySp^{(i)}f^{(i)} = Sa^{(i)}f^{(i)} + Sm^{(i)}f^{(i)}\gamma^{(i)} + Sn^{(i)}f^{(i)}\lambda^{(i)} + \dots,$$

By multiplying next the equation of condition by another system $g^{(i)}$ of factors, the reunion of the products will give a second final equation

$$xSl^{(i)}g^{(i)} + ySp^{(i)}g^{(i)} = Sa^{(i)}g^{(i)} + Sm^{(i)}g^{(i)}\gamma^{(i)} + \dots.$$

One draws from these two final equations

$$x = \frac{Sa^{(i)}f^{(i)}Sp^{(i)}g^{(i)} - Sa^{(i)}g^{(i)}Sp^{(i)}f^{(i)}}{L} + \frac{Sm^{(i)}f^{(i)}\gamma^{(i)}Sp^{(i)}g^{(i)} - Sm^{(i)}g^{(i)}\gamma^{(i)}Sp^{(i)}f^{(i)} + \dots}{L},$$

L being equal to

$$Sl^{(i)}f^{(i)}Sp^{(i)}g^{(i)} - Sl^{(i)}g^{(i)}Sp^{(i)}f^{(i)}.$$

The coefficient of $\gamma^{(i)}$ in this value is

$$\frac{m^{(i)}f^{(i)}Sp^{(i)}g^{(i)} - m^{(i)}g^{(i)}Sp^{(i)}f^{(i)}}{L}.$$

By changing $m^{(i)}$ into $n^{(i)}$, $r^{(i)}$, \dots , one will have the coefficients corresponding to $\lambda^{(i)}$, $\delta^{(i)}$, \dots . By naming therefore s the value of the part of x dependent on the errors $\gamma^{(i)}$, $\lambda^{(i)}$, $\delta^{(i)}$, \dots , the probability of this value will be, by that which precedes, proportional to the exponential

$$e^{-\frac{s^2}{H}},$$

by making

$$H = \frac{SM^{(i)}f^{(i)2}(Sp^{(i)}g^{(i)})^2 - 2SM^{(i)}f^{(i)}g^{(i)}Sp^{(i)}f^{(i)}Sp^{(i)}g^{(i)} + SM^{(i)}g^{(i)2}(Sp^{(i)}f^{(i)})^2}{L^2},$$

$M^{(i)}$ being equal to

$$\frac{4k''}{k}m^{(i)2} + \frac{4\bar{k}''}{k}n^{(i)2} + \frac{4\bar{\bar{k}}''}{k}r^{(i)2} + \dots$$

It is necessary now to determine $f^{(i)}$ and $g^{(i)}$ in a manner that H is a minimum. For this, one will make $f^{(i)}$ vary, and one will equate to zero the coefficient of its differential; this which will give, by naming P the numerator of the expression of H ,

$$\begin{aligned}
0 = & M^{(i)} f^{(i)} (\mathbf{S} p^{(i)} g^{(i)})^2 - M^{(i)} g^{(i)} \mathbf{S} p^{(i)} f^{(i)} \mathbf{S} p^{(i)} g^{(i)} \\
& - p^{(i)} \mathbf{S} M^{(i)} f^{(i)} g^{(i)} \mathbf{S} p^{(i)} g^{(i)} + p^{(i)} \mathbf{S} M^{(i)} g^{(i)2} \mathbf{S} p^{(i)} f^{(i)} \\
& - \frac{P}{L} (l^{(i)} \mathbf{S} p^{(i)} g^{(i)} - p^{(i)} \mathbf{S} l^{(i)} g^{(i)})^2.
\end{aligned}$$

It is easy to see that one satisfies this equation by supposing

$$f^{(i)} = \frac{l^{(i)}}{M^{(i)}}, \quad g^{(i)} = \frac{p^{(i)}}{M^{(i)}};$$

and one must conclude from it that one would satisfy, by the same supposition, the corresponding equation that would give $dH = 0$, by making $g^{(i)}$ vary. One sees that the same values of $f^{(i)}$ and $g^{(i)}$ satisfy the similar equations which result from the consideration of the element y .

If one has, among the elements x, y, z, \dots , some equations of condition represented by the general equation

$$l^{(i)} x + p^{(i)} y + q^{(i)} z + \dots = a^{(i)} + m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + r^{(i)} \delta^{(i)} + \dots,$$

$\gamma^{(i)}, \lambda^{(i)}, \delta^{(i)}, \dots$ being the errors of diverse kinds, one will find by the preceding analysis that the factors by which one must multiply respectively this equation, in order to form the final equations which give the values of the most advantageous elements, are, for the first final equation, represented by

$$\frac{l^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{k} n^{(i)2} + \frac{\bar{\bar{k}}''}{k} r^{(i)2} + \dots}$$

They are represented, for the second final equation, by

$$\frac{m^{(i)}}{\frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{k} n^{(i)2} + \frac{\bar{\bar{k}}''}{k} r^{(i)2} + \dots}$$

and thus consecutively. By applying therefore to the equations thus multiplied the analysis of n° 2 of the first Supplement, one will have the values of the most advantageous elements and the laws of probabilities of their errors.

In order to give an example of this application, we consider only two elements x and y . If one makes

$$M^{(i)} = \frac{k''}{k} m^{(i)2} + \frac{\bar{k}''}{\bar{k}} n^{(i)2} + \frac{\bar{\bar{k}}''}{\bar{\bar{k}}} r^{(i)2} + \dots$$

One will multiply the previous equation of condition by $\frac{p^{(i)}}{M^{(i)}}$, and one will draw from it

$$xS \frac{l^{(i)} p^{(i)}}{M^{(i)}} + yS \frac{p^{(i)2}}{M^{(i)}} = S \frac{a^{(i)} p^{(i)}}{M^{(i)}} + S \frac{p^{(i)}}{M^{(i)}} (m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + \dots);$$

but the condition of the most advantageous method gives

$$0 = S \frac{l^{(i)}}{M^{(i)}} (m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + \dots),$$

$$0 = S \frac{p^{(i)}}{M^{(i)}} (m^{(i)} \gamma^{(i)} + n^{(i)} \lambda^{(i)} + \dots);$$

one will have therefore

$$y = \frac{S \frac{a^{(i)} p^{(i)}}{M^{(i)}} - xS \frac{p^{(i)} l^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}}.$$

Substituting this value of y into the general equation of condition, and making

$$l_1^{(i)} = l^{(i)} - p^{(i)} \frac{S \frac{l^{(i)} p^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}},$$

$$a_1^{(i)} = a^{(i)} - p^{(i)} \frac{S \frac{l^{(i)} p^{(i)}}{M^{(i)}}}{S \frac{p^{(i)2}}{M^{(i)}}},$$

one will have

$$x = \frac{S \frac{a_1^{(i)} l_1^{(i)}}{M^{(i)}}}{S \frac{l_1^{(i)2}}{M^{(i)}}};$$

and the probability of an error s of this value will be proportional to

$$\frac{s^2}{c} S \frac{l_1^{(i)2}}{M^{(i)}}.$$

This analysis supposes knowledge of the constants $\frac{k''}{k}$ and $\frac{\bar{k}''}{\bar{k}}$. But one is able to obtain, by the same observations, some very close values, in the following manner.

We imagine that one has determined the elements x, y, z, \dots by the method according to which one forms the final equations, by multiplying each equation of condition successively by the corresponding coefficient of each element. If one substitutes the values of the elements thus determined in the equation of condition

$$l^{(i)}x + p^{(i)}y + \dots - a^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots,$$

one will have an equation of this form

$$R^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)} + \dots$$

We suppose, for greater simplicity, that one has only two kinds of errors $\gamma^{(i)}$ and $\lambda^{(i)}$: one will multiply first the preceding equation by $m^{(i)}$. By elevating next each member to the square and taking the sum of all the equations thus formed, one will have

$$Sm^{(i)2}R^{(i)2} = S(m^{(i)4}\gamma^{(i)2} + 2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)} + n^{(i)2}m^{(i)2}\lambda^{(i)2}).$$

The mean value of $m^{(i)4}\gamma^{(i)2}$ is evidently

$$\frac{m^{(i)4} \int \gamma^2 d\gamma \phi(\gamma)}{\int d\gamma \phi(\gamma)},$$

the integrals being taken from $\gamma = -\infty$ to γ infinity, this which gives $\frac{2k''m^{(i)4}}{k}$. One has similarly $\frac{2\bar{k}''}{k}m^{(i)2}n^{(i)2}$ for the mean value of $m^{(i)2}n^{(i)2}\lambda^{(i)2}$. One finds in the same manner that the mean value of $2m^{(i)3}n^{(i)}\gamma^{(i)}\lambda^{(i)}$ is null; one has therefore, by substituting instead of the quantities their mean values, that which one is able to make with so much more precision as the number of observations is greater,

$$Sm^{(i)2}R^{(i)2} = \frac{2k''}{k}m^{(i)4} + \frac{2\bar{k}''}{k}m^{(i)2}n^{(i)2}.$$

One will have similarly

$$Sn^{(i)2}R^{(i)2} = \frac{2k''}{k}m^{(i)2}n^{(i)2} + \frac{2\bar{k}''}{\bar{k}}n^{(i)4};$$

whence one draws

$$\frac{4k''}{k} = \frac{2Sn^{(i)4}Sm^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}Sn^{(i)2}R^{(i)2}}{Sm^{(i)4}Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2},$$

$$\frac{4\bar{k}''}{\bar{k}} = \frac{2Sm^{(i)4}Sn^{(i)2}R^{(i)2} - 2Sm^{(i)2}n^{(i)2}Sm^{(i)2}R^{(i)2}}{Sm^{(i)4}Sn^{(i)4} - (Sm^{(i)2}n^{(i)2})^2};$$

be designating therefore by $2P$ and $2Q$ the numerators of these two expressions, the factors by which one must multiply the equation of condition will be

$$\frac{l^{(i)}}{m^{(i)2}P + n^{(i)2}Q},$$

$$\frac{p^{(i)}}{m^{(i)2}P + n^{(i)2}Q},$$

.....

The concern now is to show that these values of $\frac{4k''}{k}$, $\frac{4\bar{k}''}{\bar{k}}$ are very close. For this, we will consider only one element x : the equation of condition

$$l^{(i)}x = a^{(i)} + m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)}$$

will give

$$x = \frac{Sa^{(i)}l^{(i)}}{Sl^{(i)2}} + \frac{Sl^{(i)}m^{(i)}\gamma^{(i)} + Sl^{(i)}n^{(i)}\lambda^{(i)}}{Sl^{(i)2}}.$$

Substituting this value in the equation of condition, one will have

$$R^{(i)} = \frac{l^{(i)}Sa^{(i)}l^{(i)} - a^{(i)}Sl^{(i)2}}{Sl^{(i)2}},$$

$$R^{(i)} + l^{(i)}\frac{Sl^{(i)}m^{(i)}\gamma^{(i)} + l^{(i)}n^{(i)}\lambda^{(i)}}{Sl^{(i)2}} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)};$$

but it is easy to see that the values of $Sl^{(i)}m^{(i)}\gamma^{(i)}$ and of $Sl^{(i)}n^{(i)}\lambda^{(i)}$ are nulls by the supposition of the negative errors as probable as the positive errors: one is able therefore to make, as above,

$$R^{(i)} = m^{(i)}\gamma^{(i)} + n^{(i)}\lambda^{(i)},$$

this which it was necessary to establish.

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