We have seen how compound interest works; as an investment method, it is far preferable to simple interest. Simple interest investments grow linearly, since the interest payment is a certain percentage of the initial investment only; the account grows by earning the same amount at each payment period. Compound interest investments grow exponentially since the interest payment is a certain percentage of what sits in the account at each payment period, which includes previously earned interest. Therefore, since exponential functions are concave up, compound interest investments grow faster and faster over time, making them much more attractive tools for earning money.

Suppose that a certain principal amount $P$ is being invested for $t$ years at an annual interest rate $r$, compounded $n$ times per year (at regular intervals). Then, as we explain below, the amount $A$ in the account after $t$ years pass is given by the compound interest formula:

$$A = P \left(1 + \frac{r}{n}\right)^{nt}$$

Notice that this has the form of an exponential function $A = f(t) = a \cdot b^t$ with $a = P$ and $b = \left(1 + \frac{r}{n}\right)^n$. It is easy to see that $a = P$ represents the initial output of the function, the amount at time $t = 0$. The formula for the growth factor $b$ indicates that $n$ times during the course of a year, we must multiply by the current amount by the factor $1 + \frac{r}{n}$, since at each of these $n$ compounding periods we must award one $\frac{r}{n}$th of the year’s interest $r$; in other words, the account must grow by the percentage $\frac{r}{n}$ in $n$ separate stages over the course of the year. So after a year is completed, the previous year’s balance has grown by a factor of $b = \left(1 + \frac{r}{n}\right)^n$. 
1. Now if compounding is good for investments, compounding more frequently is even better. Investors can squeeze the most out of their principal by compounding more and more times over the course of a year. Use the compound interest formula to compare the balances in a 5-year investment of $3000 at 6.5% annual interest compounded

(a) annually

(b) quarterly

(c) monthly

(d) daily (assuming a 365-day year)

(e) once every second

(f) Discuss the benefits of increasing the number of compounding periods here to even larger amounts (like \( n = \) ten million, for example).

2. As we have seen, compounding more frequently increases the balance in a compound interest investment without changing its interest rate \( r \), also called the nominal rate for the investment. The more frequent compounding has the effect of boosting the interest rate to a larger value. The effective rate or yield, of the investment, denoted \( r_{\text{eff}} \), is the rate that would yield the same balance with annual compounding that the investment produces when we compound more frequently. In other words, using the formula for compound interest,

\[
P(1 + r_{\text{eff}})^t = P \left( 1 + \frac{r}{n} \right)^{nt}.
\]

Dividing by \( P \) and setting \( t = 1 \) in this equation gives the simpler formula

\[
1 + r_{\text{eff}} = \left( 1 + \frac{r}{n} \right)^n
\]

Use this to determine the effective yields corresponding to each of the investments in #1 above.
3. Note from the last formula we cite that the effective rate of an investment does not depend on how much we invest (the principal $P$) or how long ($t$), but only on the nominal rate and the number of compounding periods. Find the effective rates (to the nearest hundredth of a percent) for the following investments:

(a) nominal rate of 2.75%, compounded quarterly (a typical CD)

(b) nominal rate of 19.99%, compounded daily (typical finance charge on a credit card)

4. We observed in problem #1 above that, while increasing the number $n$ of compounding periods of an investment of a certain principal $P$ at a nominal rate $r$ for $t$ years does increase the balance of the account, there is a limit to the benefit this achieves. Let us explore this limiting behavior.

(a) At a nominal rate of 2% ($r = 0.02$), what is the effective yield $r_{eff}$ when we compound continuously, that is infinitely often? (As we saw earlier, increasing the number of compounding periods from once a second – which is more than 3 million times a year – to $n = 10,000,000$ has a negligible effect on the final balance. Therefore, it is possible to answer this question by treating $n = \infty$ as though it were $n = 10,000,000$, or some other very large number.)

(b) For each of the nominal rates $r = 0.03, 0.04, 0.05, \ldots, 0.12, 0.13$, determine the corresponding continuous effective yields $r_{eff}$. Therefore, $b_{eff} = 1 + r_{eff}$ is the appropriate growth factor to use when computing balances on continuous compound interest investments at these nominal rates. Build a two-column table that associates to each nominal rate $r$ its continuous effective growth factor $b_{eff}$.

(c) Treating the table you assembled in (b) as defining a function $b_{eff} = f(r)$ that associates to a nominal interest rate $r$ its corresponding continuous effective growth factor, let’s try to determine what sort of function it is. By viewing a scatterplot of this data, what kind of function does it appear that $b_{eff} = f(r)$ is?
(d) Calculate average rates of change between consecutive values of the function \( b_{\text{eff}} = f(r) \) in your table. What does this analysis tell you about the concavity of the function? Does it suggest that the function is linear?

(e) Find the exponential regression function that models the data from the table, record your parameters to three decimal places and compare with the value of the special number \( e = 2.7182818 \ldots \). Build a table of values of your regression function and compare it to the table you produced in (b). Does this suggest that \( b_{\text{eff}} = e^r \)?

(f) The conclusion in (e), combined with the compound interest formula, implies that the formula for continuous compound interest is

\[
A = P(b_{\text{eff}})^t = Pe^{rt}.
\]

Use this formula to find the balance in a 5-year investment of $3000 at 6.5% annual interest compounded continuously, and compare your answer with what you found in #1 above.

5. Which is the better investment, one that pays 12.25% compounded quarterly, or one that pays 12.10% compounded continuously?