Formulas for linear functions

The simplest functions, from the perspective of formulas, are **linear functions**. This term is used for functions whose graphs are straight lines.

Since a line is completely determined by any two points along it, we ought to have full information about a linear function if we known the coordinates of any pair of points on the line. If (x_1, y_1) and (x_2, y_2) are any two points on a given line, then we can calculate the slope of the line via

$$m = \frac{y_2 - y_1}{x_2 - x_1}$$

The slope measures the rate of increase or decrease along the line, or the degree of inclination of the line with the horizontal. *This measure will not depend on which pair of points on the line are used* to make the computation because the line is straight. It follows that if (x, y) is *any* other point on the line, we will compute the same slope by pairing (x_1, y_1) with (x, y):

$$m = \frac{y - y_1}{x - x_1}$$

Solving for y in this equation will give a formula for the linear function:

$$y - y_1 = m(x - x_1)$$
$$\downarrow$$
$$\boxed{y = m(x - x_1) + y_1}$$

This last equation is known as the **point-slope form** of the linear function. Continuing with the algebra,

$$\begin{aligned} y &= mx - mx_1 + y_1 \\ &= mx + (y_1 - mx_1) \end{aligned}$$

and if we set $b = y_1 - mx_1$, this yields the formula

$$\boxed{y = mx + b,}$$

the **slope-intercept** form of the line. The parameter b is called the y-**intercept** of the line for the following reason: setting $x = 0$ in the equation yields the output $y = b$ and it corresponds to looking where the graph of the function crosses the y-axis.

Notice that vertical lines will fail the vertical line test, hence are not graphs of linear functions. Vertical lines are the only lines which *have no slope* (compare with horizontal lines, which have 0 slope).

How then are we to give equations that represent vertical lines? Such lines will have equations of the form $x = x_1$.

<u>All</u> lines can be represented in the **standard form**

$$Ax + By + C = 0$$

as the following analysis shows: If the line has a slope, then we know it can be written in the form $y = mx + b$, but then we can move all terms to the left side of the equation to get

$$-mx + y - b = 0 \quad (A = -m, B = 1, C = -b);$$

and if the line has no slope, it has the form

$$x - x_1 = 0 \quad (A = 1, B = 0, C = -x_1).$$

In applications, if we know that the functional relation between the variables is *characterized by a constant rate of change*, then the underlying function is linear and its slope measures this rate of change. This information can be exploited to write down a formula for the function.

Different lines with equal slope will have the same inclination with the horizontal, hence they will be **parallel**.

Lines which are perpendicular will have different slopes. In fact, take the intersection point P of the perpendicular lines l and L and some other point Q on l; let R be the point with the same y-coordinate as P and the same x-coordinate as Q, and let S be the point on L with the same x-coordinate as Q (Figure 1.41, p. 40). It follows from the similarity of the triangles PQR and PRS that corresponding parts of the triangles are in proportion with each other. We deduce that the **perpendicular** lines l and L have slopes m and M which are **negative reciprocals** of each other:

$$M = -\frac{1}{m}.$$