Quadratic functions

One of the simplest kinds of functions that exhibit concavity are **quadratic functions**, so called because their formulas involve the square of the input variable. The standard form of a quadratic function has formula

$$y = ax^2 + bx + c$$

where the parameters a, b, and c can be any fixed real numbers (except that a cannot equal 0 — else the function would be linear and not quadratic).

The graph of a quadratic function has a characteristic curved shape, called a **parabola**. A parabola has a fixed concavity (always up, or always down) across its entire domain.

If $a > 0$ (a is called the **leading coefficient**), then *the curve is concave up everywhere* and so the function must eventually be increasing for values of x larger than some number; also, the function must be decreasing on some interval of inputs below some number. It follows that there is a unique point along the curve where the function takes its *minimum* value; this point (h, k) is called the **vertex** of the curve.

In an entirely similar manner, if $a < 0$, then *the parabola is concave down eveywhere*, and the function must eventually be decreasing for values of x larger than some number and increasing for values of x smaller than some number. It follows that there is a unique point on the curve where the function takes its *maximum* value; this point (h, k) is also called the **vertex** of the curve.

Note that the curve is **symmetric** on either side of the vertical line $x = h$ through the vertex. That is, values of the function at points equally spaced on either side of $x = h$ must be the same: for any number n, $f(h+n) = f(h-n)$. Consequently, the graph of a quadratic function is a parabola that must have either 0, 1 or 2 x-intercepts, depending on where the vertex of the curve lies relative to the x-axis.

If the curve is concave up and the vertex lies above the x-axis, then there are no x-intercepts; if the vertex lies on the x-axis, then it is the only x-intercept; and if the vertex lies below the x-axis, then there are two x-intercepts.
If the curve is concave down and the vertex lies above the x-axis, then there are two x-intercepts; if the vertex lies on the x-axis, then it is the only x-intercept; and if the vertex lies below the x-axis, then there are zero x-intercepts.

These x-intercepts, also called the **roots** of the function, correspond to the zero, one, or two solutions to the equation $ax^2 + bx + c = 0$.

These quadratic equations can sometimes be solved by **factoring**:

$$a(x^2 + \frac{b}{a}x + \frac{c}{a}) = 0$$
$$a(x - r)(x - s) = 0$$

If this is possible, the underlying quadratic function can be written in **factored form** as

$$y = a(x - r)(x - s).$$

Once the expression is factored, the equation can be easily solved: since a product equals zero only when one of the factors is zero, and $a \neq 0$, we must have that either

$$x - r = 0 \quad \text{or} \quad x - s = 0$$

It follows that the roots of the function occur at $x = r$ or $x = s$. If r and s have different values, this corresponds to the case where the quadratic function has two roots; if $r = s$, then we are in the case in which the quadratic function has one root.

If the equation cannot be factored easily, we can still find the roots via **the quadratic formula**:

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The '±' sign in the formula indicates that the formula may produce two roots. The expression under the radical sign, $b^2 - 4ac$, is called the **discriminant** of the quadratic since it "discriminates" amongst the three possible cases. If the discriminant is positive, then $x = \frac{-b + \sqrt{b^2 - 4ac}}{2a}$ and $x = \frac{-b - \sqrt{b^2 - 4ac}}{2a}$ are two distinct roots of the quadratic. If the discriminant equals 0, then the formula produces only one root, $x = -\frac{b}{2a}$ (which would necessarily correspond to the vertex of the parabola). If the discriminant is negative, then since negative numbers do not have square roots, there are no roots at all.