Exponential Functions

Linear functions are used to describe quantities that grow (or decline) at a constant rate. Quadratic functions are used to describe quantities whose rates of growth grow (or decline) at constant rates. But there is a commonly occurring class of phenomena in which the important quantities grow in a manner different from either of these.

*Example:* Some **population growth models** are based on a simple assumption: the number of births at any given moment in time is determined by the (essentially) fixed percentage of the population representing those females who are giving birth. That is, *population grows at a fixed percentage rate of change.* For instance, Example 2, p. 107, describes how the population of Mexico increased at a constant annual percentage rate of 2.6% per year during the 1980s.

*Example:* Many financial investment instruments grow via **compound interest** at constant percentage interest rates. See p. 112, #20.

*Example:* **Radioactive decay** arises when a fixed percentage of the atoms in a substance spontaneously break down into smaller components, releasing significant amounts of energy in the process. See Example 3, p. 107-108.
Numerical investigation of these different kinds of quantities display similar behaviors: an initial amount \( a \) of the quantity grows (or decays) at a fixed percentage rate of change \( r \) with time \( t \) (initial time corresponding to \( t = 0 \)). Thus,

the amount at time \( t = 1 \) is \( a(1+r) \),
the amount at time \( t = 2 \) is \( [a(1+r)](1+r) = a(1+r)^2 \),
the amount at time \( t = 3 \) is \( [a(1+r)^2](1+r) = a(1+r)^3 \),

and so on. It follows that the function for the amount present at any time \( t \) is given by the formula \( f(t) = a(1+r)^t \), or, if we put \( b = 1+r \), the formula is more simply given as

\[
f(t) = ab^t.
\]

The number \( b = 1+r \) is called the growth factor (or decay factor); when the quantity grows, \( r \) is positive and \( b > 1 \), and when the quantity decays, \( r \) is negative and \( b < 1 \).

As the formula shows the input variable \( t \) in the exponent, the function \( f \) above is called an exponential function. Exponential functions are used to model phenomena that grow (or decay) at constant percentage rates.