Compound Interest and the Number $e$

Financial institutions offer investment instruments that compound interest (that is, fold the interest earned on the principal invested back into the account so as to earn more interest) over various intervals of time. As we have seen, the size of such an investment grows exponentially over time.

If we let $B(t)$ represent the balance in the account (in dollars) after $t$ years of investment when we invest a principal of $P$ dollars, then

$$B(t) = P(1 + r)^t$$

where $r$ is the annual interest rate. This describes an account in which interest is compounded annually: interest is credited to the account only at the end of each year.

Compare this to the situation of an account that credits interest compounded monthly, weekly, daily, or even once a minute.

*Example: p. 139, #8(a-c)*

We see that the account balance increases as the number of annual compounding periods increases, even though the principal and interest remain the same.
In general, if there are \( n \) annual compounding periods, the balance is computed via the **compound interest formula**

\[
B(t) = P \left(1 + \frac{r}{n}\right)^{nt}
\]

*Example: p. 139, #3.*

Notice that the balance cannot grow arbitrarily large *only* by increasing the number of annual compounding periods.

Note that in the formula for \( B(t) \), the growth factor has the form

\[
b = \left(1 + \frac{r}{n}\right)^n
\]

(so that we may write the formula in the simpler form \( B(t) = Pb^t \)). Only the growth factor is affected by increasing the number of compounding periods. By studying what happens to the value of \( b \) (for a variety of interest rates \( r \)) as \( n \to \infty \), we see that there is a limit to the benefit of compounding more frequently.
By setting $n$ to be a very large number (like $n = 1,000,000$), we see that $b$ itself grows exponentially with $r!$ We can therefore view $b$ as a function of $r$ with formula $b(r) = \alpha \beta^r$.

Since $b(0) = \alpha$ would be the growth factor corresponding to 0% interest, we must have $\alpha = 1$, so that $b(r) = \beta^r$. Finally, letting $r = 1$, we can determine numerically that $b(1) = \beta \approx 2.718$. In fact, this number is a very important number in higher mathematics; it was first studied extensively by the eighteenth century Swiss mathematician Leonhard Euler and has been labeled $e$ in his honor. More accurately,

$$e = 2.71828182\ldots$$

is an irrational number (like $\pi$). Incorporating this into the formulas we now have shows that $b(r) = e^r$ and so $B(t) = Pb^t = P(e^r)^t = Pe^{rt}$. 

Consequently, it makes sense to define the notion of **continuous compounding**, whereby the interest is compounded *at every moment in time* (i.e., infinitely often). As we have just found, the formula for continuous compounding has the form

\[ B(t) = Pe^{rt} \]

for principal \( P \) and interest rate \( r \). Note the difference between this and the standard compound interest formula, which involves a *finite* number of compounding periods \( n \).

*Example: p. 139, #8(d), 19.*
Nominal and Effective Percentage Rates

Interest-bearing investments are characterized by their interest rates. As we have discussed above, the growth of such an investment depends heavily on its **nominal rate** of interest \( r \) (which is typically an annual rate). But if interest is compounded, then the **effective rate** of interest \( R \) (also called the **annual percentage yield**), measures the **actual** annual percentage growth of the account, and is somewhat higher than the nominal rate.

In the two compound interest formulas, we have

\[
B(t) = P \left(1 + \frac{r}{n}\right)^{nt} \quad \text{or} \quad B(t) = Pe^{rt}.
\]

If we write them in the form \( B(t) = Pb^t \), then since the growth factor is related to the effective yield by the simple relation \( b = 1 + R \), it follows that

\[
1 + R = \left(1 + \frac{r}{n}\right)^n \quad \text{or} \quad 1 + R = e^r.
\]

*Example:* p. 139, #13.