Logarithms

It is not too hard to see that the inverse function of any linear function (with nonzero slope) is another linear function.

Things are not as straightforward for exponential functions: the inverse of an exponential function cannot be another exponential function. (One way to see this is to consider the domain and range: the domain of an exponential function is the set of all real numbers while its range consists of only positive real numbers.) The inverse function of an exponential function is called a **logarithmic** function, and has as domain only positive real numbers; its range is the set of all reals.

To describe what a logarithmic function is, we must first consider the fundamental idea of a **logarithm**. Consider the simple exponential function $y = 10^x$; here is a partial table of values:

x	0	1	2	3	4	5
y	1	10	100	1000	10000	100000

The inverse of this function reverses the roles of x and y and defines the **common logarithm function** $y = \log x$; it has this table of values:

x	1	10	100	1000	10000	100000
y	0	1	2	3	4	5

We can interpret this function by saying that $y = \log x$ is *the exponent required to express x as a power of* 10. This definiton extends to any positive number x. For instance, $\log 30 \approx 1.477$ because $30 \approx 10^{1.477}$; the `log` button on your calculator can compute a more accurate value of log30. (Since 30 lies between 10 and 100, it stands to reason that log30 lies between $\log 10 = 1$ and $\log 100 = 2$.)

- For any number x, we have $\log(10^x) = x$.
- For any positive number x, we have $10^{\log x} = x$.
- $\log 1 = 0$ and $\log 10 = 1$.
- Numbers larger than 1 have positive logarithms and numbers between 0 and 1 have negative logarithms.

[*Examples*: p. 157, #2, 7, 19]

The most important properties of logarithms are those that follow from the fundamental rules of arithmetic with exponents. Suppose that a and b are positive numbers. Then since $a = 10^{\log a}$ and $b = 10^{\log b}$, it follows that likewise $10^{\log(ab)} = ab = 10^{\log a} \cdot 10^{\log b} = 10^{\log a + \log b}$, and so

$$\boxed{\log(ab) = \log a + \log b}$$

similarly, $10^{\log(a/b)} = \dfrac{a}{b} = \dfrac{10^{\log a}}{10^{\log b}} = 10^{\log a - \log b}$, so

$$\boxed{\log\left(\frac{a}{b}\right) = \log a - \log b}$$

and since $10^{\log(b^t)} = b^t = (10^{\log b})^t = 10^{t \cdot \log b}$, we also have that

$$\boxed{\log(b^t) = t \cdot \log b}$$

[*Examples*: p. 157, #26]

In a completely analogous manner, we can use the inverse of the exponential function $y = e^x$ with special base *e* to define a different type of logarithm function, the **natural logarithm function** $y = \ln x$: $\ln x$ is *the exponent required to express the number x as a power of e.*

[*Examples*: p. 157, #3, 8, 20]

The natural logarithm has properties entirely similar to those of the common logarithm:

- for all *x*, $\ln(e^x) = x$;
- for all positive *x*, $e^{\ln x} = x$;
- $\ln 1 = 0$ and $\ln e = 1$;
- $\ln(ab) = \ln a + \ln b$;
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$;
- $\ln(b^t) = t \cdot \ln b$.

See p. 156 for a helpful guide to typical errors in using logarithms.

[*Examples*: p. 158, #30]

Logarithms are handy for solving certain kinds of **exponential equations**, those in which the variable appears in the exponent.

[*Examples*: p. 157-159, #15, 36; #14, 44, 46]