Quadratic Functions Revisited

When we first studied quadratic functions, we saw that their graphs are always parabolas and their formulas can be expressed in **standard form** as $y = f(x) = ax^2 + bx + c$, and (sometimes) in **factored form** as $y = f(x) = a(x - r_1)(x - r_2)$.

We were able to make a connection between the values of the parameters *a*, *b*, and *c* and the graph of the function:

- the sign of *a* determines whether the parabola opens up (*a* positive) or down (*a* negative);
- *b* is helps to determine the position of the vertex of the parabola, which falls on the axis of symmetry at $x = -\dfrac{b}{2a}$; and
- *c* is the *y*-intercept.

In addition, the factored form identifies the two roots or zeros, $x = r_1, r_2$ of the function; these are also the *x*-intercepts of the graph.

[Examples: p. 231, #18, 6]

We can now extend these concepts by using the idea of function transformations.

The simplest quadratic function is the prototypical function $y = f(x) = x^2$. A vertical stretch by the factor a produces from it the transformed function $y = f_1(x) = ax^2$. To transform the parabola so that its vertex moves from the origin to the point (h, k), we perform a horizontal shift to the right h units (that is, to the left $-h$ units) to obtain $y = f_2(x) = a(x-h)^2$, followed by a vertical shift up k units to obtain $y = f_3(x) = a(x-h)^2 + k$. Therefore, every quadratic function can be characterized by its vertical stretch factor a and its vertex (h, k). We call the formula

$$y = f(x) = a(x-h)^2 + k$$

the **vertex form** of the function.

One can proceed algebraically from the standard form of a quadratic function to its vertex form by a procedure called **completing the square**:

From $f(x) = ax^2 + bx + c$, factor out the leading coefficient:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{c}{a}\right)$$

then square half of the coefficient of x and add and subtract it within the parentheses:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

(This is done so that the first three terms within the parentheses form a perfect square; this is why the procedure is called *completing the square*.)

$$\begin{aligned} f(x) &= a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\ &= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right) \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right) \end{aligned}$$

So if we let $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, we have expressed the function in vertex form:

$$f(x) = a(x-h)^2 + k.$$

[Examples: p. 231, #4, 17, 32]