Quadratic Functions Revisited

When we first studied quadratic functions, we saw that their graphs are always parabolas and their formulas can be expressed in **standard form** as 
\[ y = f(x) = ax^2 + bx + c, \] 
and (sometimes) in **factored form** as 
\[ y = f(x) = a(x - r_1)(x - r_2). \]

We were able to make a connection between the values of the parameters \( a, b, \) and \( c \) and the graph of the function:
- the sign of \( a \) determines whether the parabola opens up (\( a \) positive) or down (\( a \) negative);
- \( b \) is helps to determine the position of the vertex of the parabola, which falls on the axis of symmetry at \( x = -\frac{b}{2a} \); and
- \( c \) is the \( y \)-intercept.

In addition, the factored form identifies the two roots or zeros, \( x = r_1, r_2 \) of the function; these are also the \( x \)-intercepts of the graph.

[Examples: p. 231, #18, 6]
We can now extend these concepts by using the idea of function transformations.

The simplest quadratic function is the prototypical function \( y = f(x) = x^2 \). A vertical stretch by the factor \( a \) produces from it the transformed function \( y = f_1(x) = ax^2 \). To transform the parabola so that its vertex moves from the origin to the point \((h, k)\), we perform a horizontal shift to the right \( h \) units (that is, to the left \(-h\) units) to obtain

\[
y = f_2(x) = a(x - h)^2,
\]
followed by a vertical shift up \( k \) units to obtain \( y = f_3(x) = a(x - h)^2 + k \).

Therefore, every quadratic function can be characterized by its vertical stretch factor \( a \) and its vertex \((h, k)\). We call the formula

\[
y = f(x) = a(x - h)^2 + k
\]
the vertex form of the function.

One can proceed algebraically from the standard form of a quadratic function to its vertex form by a procedure called completing the square:

From \( f(x) = ax^2 + bx + c \), factor out the leading coefficient:

\[
f(x) = a \left( x^2 + \frac{b}{a}x + \frac{c}{a} \right)
\]
then square half of the coefficient of $x$ and add and subtract it within the parentheses:

$$f(x) = a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2} - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

(This is done so that the first three terms within the parentheses form a perfect square; this is why the procedure is called completing the square.)

$$f(x) = a\left(\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left(\left(x + \frac{b}{2a}\right)^2 - \frac{b^2}{4a^2} + \frac{c}{a}\right)$$

$$= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right)$$

So if we let $h = -\frac{b}{2a}$ and $k = c - \frac{b^2}{4a}$, we have expressed the function in vertex form:

$$f(x) = a(x - h)^2 + k.$$ 

[Examples: p. 231, #4, 17, 32]