Composition of Functions and their Inverses

Functions are types of relationships between pairs of quantities, an input and an output. But there are often situations in which functions can link pairs of variables in a chain of relationships involving three or more quantities. (See Example 1, p. 354)

When the output of a function g is used as the input to another function f, we can talk about a third function, the **composition** of f with g. If g has input t and output $g(t)$, and f takes $g(t)$ as input, its output is $f(g(t))$. *The composition function has input t and output* $f(g(t))$.

[*Example*: p. 359, #1, 19]

Formulas for composite functions are found by substituting the expression for the output of the first "inner" function into the input variable in the formula for the second "outer" function.

[*Example*: p. 359, #5, 9, 19, 39]

A more challenging problem, but very useful in some cases, is **decomposing** a function: expressing it as the composition of two simpler functions.

[*Examples*: p. 361, #49, 57]

Recall that if f is a function such that $y = f(x)$ and that *each y value occurs only once as the output of some x*, then f has an **inverse function**, called f^{-1}, which takes the outputs y of f as inputs and returns the corresponding input x from f as its output. Clearly, this means that the domain and range of f are reversed to obtain the domain and range of f^{-1}.

Observe that for every value of x and y,

$$f^{-1}(f(x)) = x \text{ and } f(f^{-1}(y)) = y.$$

That is, the composition of a function and its inverse, in either direction, represents the **identity function** (the function that takes every input to itself as output).

[*Example*: p. 370, #18, 22]

The existence of an inverse function to *f* requires that each output value occur only once as the output for some input. Graphically this means that *every horizontal line that meets the graph of f must meet it only once*: this is the **horizontal line test** for existence of inverses.

Further, the graph of *f* and its inverse f^{-1} are closely related: for every point (x, y) on the graph of *f*, the corresponding point (y, x) must lie on the graph of f^{-1}. Therefore, *the graph of* f^{-1} *is obtained from that of f by reflection across the line* $y = x$.

Some functions do not have inverses (because their graphs do not pass the horizontal line test). However, it may be possible to modify the function by **restricting its domain** to a smaller set of numbers for which it does pass the horizontal line test. For instance, $f(x) = x^2$ does not pass the horizontal line test when its domain is the set of all real numbers, but if we restrict the domain to $x \geq 0$, then it does have an inverse, namely the function $f^{-1}(x) = \sqrt{x}$.

[*Example*: p. 370, #10, 30, 49]