Polynomial Functions

A **polynomial function of degree** n is a function which is a sum of power functions with *positive integer* powers, written in standard form as

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0;$$

each power function in this sum is a **term** of the polynomial and the constants $a_n, a_{n-1}, \ldots, a_0$ are called **coefficients**. Note that each coefficient is labeled with a subscript to identifies which power of x it is matched with. The highest power term $a_n x^n$ is the **leading term**; a_0 is the **constant term**. A polynomial of degree 0 is a constant function, a polynomial of degree 1 is a linear function, and a polynomial of degree 2 is a quadratic function. Polynomials of higher degree are called *cubic* $(n = 3)$, *quartic* $(n = 4)$, *quintic* $(n = 5)$, …

[*Examples*: p. 400, #1, 3, 5]

For sufficiently large values of x, the leading term of a polynomial function is much larger in size than all the other terms combined, so as $x \to \pm\infty$, $p(x)$ grows in the same way as $a_n x^n$. This determines the **long-term behavior** of the function.

[*Examples*: p. 400, #11]

As for **short-term behavior**, we note first that the y-intercept of the graph of $p(x)$ is its constant term a_0. (We already knew this for polynomials of degree 2 or less.) The x-intercepts are those values of x for which $p(x) = 0$. If it is possible to put the polynomial in **factored form**

$$p(x) = a_n(x - r_1)(x - r_2)\cdots(x - r_n)$$

(as we do for quadratic functions), then the roots $r_1, r_2, \ldots, r_n$ are the x-intercepts.

If the roots are listed in increasing order ($r_1 \le r_2 \le \cdots \le r_n$), then between any two distinct consecutive roots, there must be a **local maximum** or **local minimum** value of the function, namely a point x where $p(x)$ is greater than or less than all other values of p between those roots.

[*Example*: p. 406, #15]

Note that $p(x)$ may not be entirely factorable; in fact, it may not be factorable at all (as is true for quadratic functions). Thus, a polynomial of degree n can have *at most* n x-intercepts (roots). Further, a polynomial of degree n can have *at most* $n - 1$ local maximum and minimum points.

[*Examples*: p. 406, #11, 23, 42]

It may happen that roots of $p(x)$ repeat, that is, there may some equalities in the ordered list $r_1 \le r_2 \le \cdots \le r_n$ of roots of the function. If a root r repeats an odd number of times, then the graph of $p(x)$ will cross the axis there; if r repeats as a root an even number of times, then the graph of $p(x)$ will touch but not cross the x-axis at $x = r$. Whenever a root repeats, the x-axis will be *tangent* to the graph.

[*Examples*: p. 407, #27, 34, 43]