The value of a game

The saddle point payoff is called the **value** of the game, since it represents the minimum assured payoff to both the players. It represents what playing this game is worth to the players. (If both players are rational, they must play into a saddle point.)

Is it possible for the same game to have different saddle points? with different values? The answer is yes to the first question ([S, p. 9](#)), but no to the second:

**The Value Theorem.** Any two saddle points in a zero-sum game must have the same value.

**Proof.** Suppose that $a$ and $b$ are two saddle point payoffs in a matrix game. If they lie in the same row, then since $a$ is the smallest entry in its row, $a \leq b$; but since $b$ is also the smallest entry in its row, $b \leq a$. So $a = b$.

Similarly, if they lie in the same column, both are the largest entry in the column, so both $a \geq b$ and $b \geq a$, so $a = b$.

If they lie in a different row and column, then they form opposite corners of a rectangle in the matrix; let $c$ and $d$ be the payoffs in the other two corners.
Their relative positions in the matrix look like

\[
\begin{array}{ccc}
  a & \cdots & c \\
  \vdots & \ddots & \vdots \\
  d & \cdots & b \\
\end{array}
\]

Since \(a\) is the smallest entry in its row and \(b\) the largest in its column, we have \(a \leq c \leq b\). But then \(b\) is the smallest entry in its row and \(a\) the largest in its column, so \(b \leq d \leq a\). Putting these inequalities together gives \(a \leq c \leq b \leq d \leq a\). But since the same number lies at both ends of the inequality, all four numbers must in fact be equal. This shows that not only does \(a = b\), but that \(c\) and \(d\) are also saddle points.  //

An effective way to find saddle points in matrix games is to record the row minimum values to the right of the matrix and the column maximum values below. The value of the game is simultaneously the maximum value of the row minima, the \textbf{maximin}, and the minimum value of the column maxima, the \textbf{minimax}.

While a game can have many saddle points, \textit{it is possible for a game to have no saddle point at all}. In such games, any choice of pure strategy will leave at least one of the players with regret. The way to resolving this issue will be our next discussion.