Chapter 2

Pythagoras of Samos

1 The beginnings of Greek philosophy

The lyric poet Homer wrote the *Iliad* and the *Odyssey* in the seventh century BCE at a time when the city-states of Greece (Athens, Sparta, Troy, Delphi, Corinth, Thermopylae, and others) were beginning to throw off the yoke of tyrannical kings and form more cooperative governmental systems (oligarchies, ruled by aristocratic families) based more fully on principles of law rather than simply on military might. They sometimes even established rather sophisticated forms of democracy. They traded with economic partners around the Mediterranean Sea and beyond, eventually establishing colonies in other locales, like Byzantium, Massalia (Marseilles), Syracuse, and Naples. They developed a rich artistic culture remembered today for its mosaic art, sculpture, and architecture. They had imported and adapted an alphabet from their Semitic Phoenician neighbors to the east, and were using this alphabet in the creation of important works of drama and poetry. And they left a legacy of disciplined inquiry about the world in which we live – the first philosophies of the Western world.

Among these first philosophers was Thales of Miletus (ca. 640-560 BCE), who lived on the Ionian coast (what today is western Turkey). It is said that he was a successful businessman and politician. Later in life he began to espouse a belief that through rational thought and inquiry one could come to understand the underlying principles of nature, that the world obeyed laws that the human mind could comprehend, that it need not work exclusively through the capricious actions of the gods. This represents one of the first attempts at formulating a philosophy of nature, a system of rational principles that strove to explain how the world orders itself by other than supernatural means.

As a natural philosopher, Thales became a student of astronomy, almost certainly in the long tradition of Babylonian astronomers who had charted, measured and studied the motion of the visible heavenly bodies. At one point he is said to have successfully predicted a solar eclipse. He also, significantly for
our purposes, asserted a number of results in geometry, like the following:

*Angles opposite each other between any pair of intersecting straight lines are equal.*

*If two sides of a triangle are equal, then the angles opposite those sides are equal.*

*Any angle inscribed in a semicircle must be a right angle.*

What was significant about these statements was not the correctness of the results, which (in addition to being true) are compellingly believable, straightforwardly testable, and – to be frank – not altogether surprising. The real importance of Thales’ assertions was how he stated them. For perhaps the first time in history, mathematical results were being presented as *theorems*, as facts whose truths were *logically provable*. That is, they could – and should – be deduced logically through dialectic argumentation that compelled assent from all those engaged in the discourse. By arguing rationally, one could discover that these truths followed from other more basic and compelling truths like some grand sudoku puzzle. (Ironically, the historical record does not tell us what Thales’ arguments were for the theorems he stated.) Pretty soon, we find that this move to validate mathematical results through logical deduction became the accepted paradigm for doing mathematics among Greek philosophers and would later be the most enduring legacy of Greek mathematics. We find here the first glimpse of a methodology for doing mathematics that has become customary for all of its practitioners, and has dominated its practice for centuries, even to this day. It requires more than that results be correct, and much more than that they be merely useful. It requires mathematicians to always ask, "Why is this result correct, and how does it follow from other statements that have been deduced to be correct?"

One of the men who supposedly engaged Thales in discourse was the young Pythagoras of Samos (ca. 569-475 BCE). It is reported that Thales advised Pythagoras to travel to Egypt to learn more mathematics there. When, after many years, Pythagoras returned to his home on Samos, he established a school of his own called the Semicircle. But, in addition to not being favorably received back home as a teacher, his countrymen encouraged him to devote himself instead to politics. To escape these pressures, he removed himself to Croton, a town on the southern tip of the Italian peninsula, where he reestablished himself as a teacher, setting up a new school that was characteristically secretive and cultish, in the style of the Egyptians, and whose disciples were known as *mathematikoi*, literally the learned (from the Greek root verb *manthanō* = *to learn or study*: hence, *mathematics* is precisely the stuff one has to study!) At this point, the pursuit of mathematics becomes not only the focus of Pythagoras’ attention but also the center of his world view, one that, as we shall see in the text that follows, places mathematics and the idea of number at the origin of all things. This philosophy, coming as it does early on in the classical period of Greek science would help to place mathematics in a very prominent position in
the high culture of the Greek world. It would influence, directly or indirectly, nearly all subsequent Greek philosophy.

After his death, the cult that Pythagoras established flourished for at least another generation, dedicating itself ever more deeply to philosophy and especially to the study of mathematics. For centuries afterwards, many others would call themselves Pythagoreans, allying themselves with the philosophy of this school.

No writings of Pythagoras remain with us today. Indeed, what little we know of the man is shrouded in legend (as is the case for most famous figures of the ancient world). But as was common practice in the ancient world, those members of the school often promoted their work by attributing it to the famous Master of the school, thereby endowing his authority to their ideas.

One of the best sources for the work of the Pythagorean school can be found in the work of Aristotle (384-322 BCE), himself a renowned philosopher. In the first passage below, from his *Metaphysics*, Aristotle describes the world view of the Pythagoreans, a mystical mash of numbers and music that seeks order in the universe through mathematics. You should be able to tell from his tone that Aristotle is rather critical of this philosophy of nature.

The second passage below is the earliest known statement, in Euclid’s *Elements*, of the famous theorem that is attributed to Pythagoras, the one about the relation between the sides of a right triangle. As we saw in the Prologue, there is much to discover about this important result, some of which may involve un-learning what we think we already know about it!

2. Text: “All is number”

From *Metaphysics*, Aristotle (Book I, Ch. V, 985b23-986a12):

Contemporaneously with these philosophers and before them,\(^1\) the so-called Pythagoreans, who were the first to take up mathematics, not only advanced this study, but also having been brought up in it they thought its principles were the principles of all things. Since of these principles numbers are by nature the first, and in numbers they seemed to see many resemblances to the things that exist and come into being – more than in fire and earth and water\(^2\) (such and such a modification of numbers being justice, another being soul and reason, another being opportunity – and similarly almost all other things being numerically expressible\(^3\)); since, again, they saw that the modifications and the ratios of

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\(^1\)Aristotle has just listed references to a number of pre-Socratic philosophers whose cosmogonies (theories about the origin of things) he has just described: Thales, Anaximenes, Hipparus, Hercleitus, Empedocles, Anaxagoras, Leucippus, and Democritus. He continues here with a description of the cosmogony of the Pythagoreans.

\(^2\)Some of the philosophers listed in the previous note identified various elements of matter, usually including one or more of fire, earth, air, and water, which in synthesis would become the classical cosmogony, essentially the ancient Greek periodic table.

\(^3\)As one of the central tenets of Pythagorean philosophy was that the idea of number was the fundamental principle of the structure of the universe, it was important to find the numerical properties of things. Thus, all things had numbers associated with them. Here is a
the musical scales were expressible in numbers;\textsuperscript{4} – since, then, all other things

\begin{itemize}
  \item brief list of some of the attributes given to numbers by the Pythagoreans:
  \item 1: mind (from which all things have their being), the creator deity, goodness and love (as they are indivisible), form;
  \item 2: division, opposition, evil, instability, opinion, maleness, communion, marriage;
  \item 3: equilibrium (viz. the tripod stand), maleness, friendship, peace, virtue, wisdom, and harmony (these positive qualities derived from the fact that, being made from 1 and 2 together, 3 was the first odd number – being the unit against which all other numbers were measured, 1 was not considered an odd number)
  \item 4: root of all things (as it describes the nature of the tetractys, the diagram of ten symbolic dots arranged in successive rows of 1, 2, 3, and 4 dots that was the mystical emblem of their society, and also the number of points of the tetrahedron, or triangular pyramid, simplest of all solids), soul (made up of mind, science, opinion, and sense), justice;
  \item 5: health (the union of the first even number, 2, and first odd number, whence the 5-dot pentagram was a symbol of health), the ether (the fifth element);
  \item 6: totality, all of creation (the sum of the first numbers 1, 2, and 3, which are also the factors of 6);
  \item \ldots
  \item 10: perfection (the number of the tetractys), heaven, power.
\end{itemize}

\textsuperscript{4}Perhaps the most lasting discovery of the early Pythagoreans was the science of musical harmony. They found that simple number relationships were the basis for musical harmonies. Two identical strings of equal length produce the same tone when plucked. But when one is half the length of the other, the shorter one produces a tone different from, but in harmony with, the other: it produces the tone exactly one octave higher than the longer one, an interval called the diapason. Moreover, when the strings have lengths in the ratio 2:3, another harmony is produced, the diapente: the shorter string makes a tone exactly one fifth higher than the other. When the strings are in the ratio 3:4, the resulting harmony is the diatessaron, the shorter string making a tone exactly one fourth higher. The same harmonies were seen to be produced by hammers of varying weights struck upon an anvil; when the hammers have weights in the ratios 1:2, 2:3, or 3:4, the same corresponding harmonies are heard as in the plucked strings.

This discovery was evidence enough for the Pythagoreans to conclude that the origins of all things are based on numbers. They treated numbers and simple geometric designs

\begin{figure}[h]
\centering
\includegraphics[width=0.6\textwidth]{tetractys_pentagram.png}
\caption{(a) Tetractys, and (b) Pentagram}
\end{figure}
3. **Text: The Pythagorean Theorem**

From Euclid’s *Elements*, Book I, Proposition 47:

> In right-angled triangles the square on the side opposite the right angle equals the sum of the squares on the sides containing the right angle.

The difference between the Greek geometrical mind and the modern one, and it will aid greatly in the reading of the following documents to keep this in mind.

Consequently, this Pythagorean Theorem is a statement not about the relationship between the numbers which are the squares of the lengths of the sides of a right triangle, but rather about the relation between the areas of actual squares erected on the sides of the triangle (see Figure 2.2). And even this is misleading, as we moderns tend to distinguish between the square, a geometrical object, and its area, a number, while to the Greek, the number is not the feature of interest. Instead, he equates the square with the content of its area and does not ask what number this is. Numbers may be useful to describe the relative sizes of objects, but not necessarily to provide their (absolute) measurements. So the Theorem states that if one combines the squares on the two sides of any right triangle, the resulting figure would have the same size as the square on the hypotenuse.

The discussion that follows the statement of the Theorem is Euclid’s proof of the result, based on small number symmetries as talismans endowed with mystical powers or as markers of membership in the cult. These diagrams included the tetractys, a triangular diagram consisting of a perfect ten dots with four equally spaced dots per side (*tetra-* = four), and the pentagram (*penta-* = five), the configuration formed by connecting all ten possible pairs of corners of a regular pentagon.

In this cosmology, the Pythagoreans believed that the heavenly bodies all rotated about a great central fire at the hub of the universe while affixed to invisible crystalline spheres, indicative of the fact that the sphere was the most perfect of solids. (The earth moved around this fire in such a way that we never saw its burning light face on; the Sun, a mirrored sphere of glass, reflected its light to us during daytime.) These spheres moved in literal harmony with each other, the intervals between them producing musical tones, so that, for instance, the octave was produced by the harmony between the motions of the earth and the dome of the stars.

They counted (1) the earth, the seven visible planets: (2) the Moon, (3) Mercury, (4) Venus, (5) the Sun, (6) Mars, (7) Jupiter, and (8) Saturn, and (9) the celestial dome of the fixed stars (which are always seen to move in concert with each other) as the part of the universe visible from earth. These nine bodies produced the eight tones of the musical scale interspersed across one full harmonic octave. But since the number 10 was seen to be representative of perfection (see note 7 below), the Pythagoreans hypothesized an additional planet, a Counter-Earth, which was said to travel alongside the earth (while always on the
opposite side of the planet from where all humans lived, which is why it could not be seen!\).

7The perfection of the number 10 is based on the fact that the sum of the first four numbers is 10 (this is the numerical content of the tetractys diagram): $1 + 2 + 3 + 4 = 10$. Thus, 10 includes all four of the elements of nature in its essence.

8Aristotle refers to his De Caelo (On the Heavens), where he writes on astronomy.

9The first thing to note about the Pythagorean Theorem is that it probably was never stated as such by Pythagoras himself. The attribution of the result to Pythagoras was first popularized by writers like Vitruvius, Plutarch and Cicero in the centuries right at the dawn of the common era, some 500 years after Pythagoras lived. But we have already mentioned that we have nothing by way of writings, or aphorisms, or even attributions of such to the man Pythagoras. And while the school that bears his name included a number of capable mathematician/geometers, there is no reason to assert that Pythagoras was accomplished in mathematics himself. The truth of this theorem about right triangles was certainly well known long before Pythagoras, as we saw in the Epilogue. The best we can hope for is to attribute to Pythagoras not the discovery of the result, but only the first known proof of it. But here, too, we have no evidence. What is more likely is that some Pythagorean mathematician at some point between the sixth and fourth centuries BCE offered some proof of the result, but no record of who this might be or what the proof was remains. All we do know is that the earliest written proof of the theorem is found in the work we are reading here, Euclid’s Elements (of Geometry), compiled at the beginning of the fourth century BCE. In fact, Euclid gives not one but three proofs of this theorem in the Elements! (The others are found as the Lemma to Proposition X.33 and Proposition VI.17.) But observe that Euclid chose to make the Pythagorean Theorem the final result of the very first book of his grand work on geometry. (Actually, the last proposition of Book I of the Elements is the converse of the theorem; the Pythagorean Theorem is the next-to-last result.) We will return to consider the significance of this point later, in Chapter 7.

The next thing to realize about the Pythagorean Theorem in this, its Greek formulation,

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Figure 2.2: Euclid’s “Windmill”, the diagram for his proof of the Pythagorean Theorem
3. **THE PYTHAGOREAN THEOREM**

Let $ABC$ be a right-angled triangle having the angle $BAC$ right. I say that the square on $BC$ equals the sum of the squares on $BA$ and $AC$.\(^\text{10}\)

Describe the square $BDEC$ on $BC$, and the squares $GB$ and $HC$ on $BA$ and $AC$. Draw $AL$ through $A$ parallel to either $BD$ or $CE$, and join $AD$ and $FC$.\(^\text{11}\)

Since each of the angles $BAC$ and $BAG$ is right, it follows that with a straight line $BA$, and at the point $A$ on it, the two straight lines $AC$ and $AG$ not lying on the same side make the adjacent angles equal to two right angles, therefore $CA$ is in a straight line with $AG$.\(^\text{12}\)

For the same reason $BA$ is also in a straight line with $AH$.

Since the angle $DBC$ equals the angle $FBA$, for each is right, add the angle $ABC$ to each, therefore the whole angle $DBA$ equals the whole angle $FBC$.

Since $DB$ equals $BC$, and $FB$ equals $BA$, the two sides $AB$ and $BD$ equal the two sides $FB$ and $BC$ respectively, and the angle $ABD$ equals the angle $FBC$, therefore the base $AD$ equals the base $FC$, and the triangle $ABD$ equals the triangle $FBC$.\(^\text{13}\)

Now the parallelogram $BL$ is double the triangle $ABD$, for they have the

\(^{10}\)The labeling refers to the diagram of Figure 2.2.

\(^{11}\)The additional lines that are being drawn here guide the structure of Euclid’s proof. In short, what Euclid does is to argue that the square $GB$ is (in area) twice the triangle $FBC$, which is itself congruent to (hence equal in area to) triangle $ABD$. Since triangle $ABD$ is half the rectangle $BL$, he concludes that square $GB$ equals rectangle $BL$. As the same argument can be applied to the left side of the Windmill diagram, he also can prove that square $HC$ equals rectangle $CL$. But since square $BDEC$ is the sum of the two rectangles $BL$ and $CL$, the result of the Theorem follows. Details are worked out in the subsequent text.

\(^{12}\)It may seem pointless to some readers for Euclid to give an involved argument here for why $CA$ lies in the same line as $AG$; after all, it is clearly apparent from the diagram that this is so. But this brings up an important and subtle methodological point: Euclid and his contemporaries give great value to the power of logic in geometry, so much so that they will ignore the evidence of a diagram in favor of a more complicated argument. The reason for this is simple: any diagram that illustrates a geometrical theorem can only provide information about a single instance of the result. The theorem, by its nature, is a statement of generality meant to apply to (usually) an infinity of possible geometric objects. Arguing from a diagram would be fallacious (logically invalid) since it would incorrectly use the single case exemplified by the particular diagram drawn to justify something that is being argued in full generality and is meant to apply to all possible situations covered by the premises of the theorem.

\(^{13}\)Euclid is making use here of the famous side-angle-side theorem for equality of triangles (which he stated and proved early on as Proposition I.4 of the *Elements*).
same base $BD$ and are in the same parallels $BD$ and $AL$. And the square $GB$

is double the triangle $FBC$, for they again have the same base $FB$ and are in

the same parallels $FB$ and $GC$.\footnote{Proposition I.41, proved earlier by Euclid in the

Elements, states that any triangle sharing a side with a parallelogram and having its opposite vertex on the same line extending the opposite, parallel side of the parallelogram, is half the parallelogram. In a modern formulation, this would say that a triangle with base $b$ and height $h$ is half the area of a parallelogram with base $b$ and height $h$. We immediately recognize the truth of this theorem because we recall that the formula for the area of a parallelogram is $A = bh$ and that for a triangle is $A = \frac{1}{2}bh$.}

Therefore the parallelogram $BL$ also equals the square $GB$.

Similarly, if $AE$ and $BK$ are joined, the parallelogram $CL$ can also be proved
equal to the square $HC$. Therefore the whole square $BDEC$ equals the sum of

the two squares $GB$ and $HC$.

And the square $BDEC$ is described on $BC$, and the squares $GB$ and $HC$
on $BA$ and $AC$. Therefore the square on $BC$ equals the sum of the squares on

$BA$ and $AC$.

Therefore in right-angled triangles the square on the side opposite the right

angle equals the sum of the squares on the sides containing the right angle.
Q.E.D.\footnote{In the formulaic writing style that Euclid employs (and which will be discussed more in

Chapters 4 and 7), the end of a proof is signaled by the words “Which was to be demon-

strated.” In the Latin translations of Euclid’s works which became widely read in the West

from the tenth century on, this was rendered “Quod erat demonstrandum,” and then often

in abbreviated form. In modern English, the abbreviation Q.E.D. is now used to signal the

completion of any sort of deductive argument.}

4 Summary

- Classical Greek philosophers emerged in the sixth century BCE who ref-

lected from human experience on how the world worked and what prin-

ciples regulated the universe and human affairs.

- Thales of Miletus began to make statements of geometrical truth and of-

ferred justifications for their veracity, thereby placing geometry on a foun-

dation of universal logical deduction rather than supernatural, or even

human, authority. From this point on, the notion of proof became an ever

more important principle in mathematics, and gave it a criterion for eval-

uating mathematical truth and authorizing the certainty of its assertions.

- The Pythagorean School adopted a philosophy that number was the found-

ing principle of the universe, buoyed by the success of their mathematical

theory of musical harmony, that harmonic sounds are characterized by

simple, small number ratios between the lengths of strings or the weights

of bells.

- The Pythagorean Theorem is actually misattributed to Pythagoras; it was

not discovered by him, nor is it likely that he gave its first proof. The first
written proof is Euclid’s *Elements*, Proposition I.47, written down at the beginning of the fourth century BCE.

- The Pythagorean Theorem in its Greek form is a geometric statement about the sizes of the three squares erected on the sides of a right triangle, not about an algebraic equation like \(a^2 + b^2 = c^2\).

Know how to...

- prove the Pythagorean Theorem as Euclid did in *Elements*, Proposition I.47.

5 Exercises

The first four exercises below present simple geometric proofs of the Pythagorean Theorem.

1. Explain how the diagram in Figure 2.3 provides a proof of the Pythagorean Theorem.

   ![Figure 2.3: A dissection proof of the Pythagorean Theorem.](image)

2. In 1830, Henry Perigal, a British bookkeeper and amateur mathematician, gave a dissection proof of the Pythagorean Theorem based on the diagram in Figure 2.4. (You should recognize Euclid’s Windmill in this diagram.) Describe how and why it works.

3. The sixteenth century Arabic geometer Thabit ibn Qurra gave a proof of the Pythagorean Theorem based on the three diagrams in Figure 2.5, reading left to right. Describe how it works.

4. The twelfth century Indian mathematician Bhaskara gave a proof of the Pythagorean Theorem that consisted of nothing more than the third diagram of Figure 2.5 with a caption containing the single word, “Behold!” Describe how it works. (Hint: move right to left across this sequence of diagrams.)
5. An American baseball diamond is in the shape of a square 90 feet to a side. What is the distance from second base to home plate (at opposite corners of this square)?

6. The Pythagorean Theorem is useful for certain types of indirect measurement which form the theoretical basis for the practice of surveying. For instance, to measure the distance across a pond, one can sight between the two locations $A$ and $B$ (see Figure 2.6), then move perpendicularly with respect to the direction to $B$ from $A$ to a point $C$ with a clear line of ground from $B$. If the distance from $A$ to $C$ is 345 feet and the distance from $C$ to $B$ is 448 feet, what is the distance from $A$ to $B$ (to the nearest foot)?

7. Euclid’s proof of Proposition I.47, made important use of his Proposition I.41:

   If a parallelogram has the same base with a triangle and is in the same parallels, then the parallelogram is double the triangle.

which in modern form says that a triangle with base $b$ and height $h$ is half the area of a parallelogram with the same base and height. We recognize the truth of this statement since we remember that the area formulas for triangle and parallelogram are, respectively, $A = \frac{1}{2}bh$ and $A = bh$. This modern view puts an algebraic interpretation on what for Euclid was an

![Figure 2.4: Perigal’s proof of the Pythagorean Theorem.](image-url)
5. EXERCISES

Figure 2.5: Thabit’s proof of the Pythagorean Theorem.

exclusively geometric statement. Give a completely geometric proof of this theorem, using the diagram of Figure 2.7. (Hint: The case illustrated on the far right is the easiest: explain why triangles $ABC$ and $CDE$ ($E = A$) are congruent. The case illustrated on the far left can be argued similarly after inserting the parallel to $CD$ through $A$ to meet $BC$ in $P$. In the middle case, use the fact that the trapezoid $ABCD$ is composed of three figures, the triangles $ABE$ and $CQD$ and the quadrilateral $BCQE$, and continue the argument from there.)

Figure 2.6: Indirect Measurement via the Pythagorean Theorem.

Figure 2.7: *Elements*, Proposition I.41 in its three cases, where the vertex $A$ of the triangle opposite the common base $BC$ with the parallelogram is between the vertices $D$ and $E$ of the opposite side (left); is not between $D$ and $E$; and coincides with one of the vertices (right).