The Fundamental Theorem of Calculus

You have probably noticed that if $f(t)$ is a rate function in the variable t, then the definite integral $\int_a^b f(t)\,dt$, which measures the accumulated change over the interval $a \le t \le b$, is measured in units whose rate of change is measured in the same units that $f(t)$ uses. More importantly, the indefinite integral function $F(x) = \int_a^x f(t)\,dt$ is measured in units whose rate of change is measured in the same units as the function $f(t)$, called the **integrand** of the integral.

This suggests a simple and important relationship between the functions $F(x)$ and $f(t)$, which is formulated as

The Fundamental Theorem of Calculus: The derivative of the indefinite integral function $F(x) = \int_a^x f(t)\,dt$ is $f(x)$. In other words,

$$\frac{d}{dx}\left[\int_a^x f(t)\,dt\right] = f(x)$$

Equivalently, this theorem states that taking a derivative (differentiation) is the inverse operation to determining the indefinite integral (integration). (This is why the thoerem is called fundamental!)

The most important consequence of the Fundamental Theorem is that to find a formula for the indefinite integral function $F(x) = \int_a^x f(t)\,dt$, we need only find a function whose derivative is $f(x)$. Any such function is also called an **antiderivative** of $f(x)$.

Notice that if $F(x)$ is an antiderivative of $f(x)$, then any modification of $F(x)$ that involves adding or subtracting a constant to the formula for $F(x)$ yields another antiderivative of $f(x)$ (since the derivative of the constant is 0, the modification does not change the fact that $F'(x) = f(x)$). Therefore, $f(x)$ will have infinitely many antiderivatives, all differing only by the choice of constant term in the formula. To illustrate this concept, we introduce a new notation: the **general antiderivative** of $f(x)$ is denoted

$$\int f(x)\,dx = F(x) + C$$

(note the absence of limiting values on the integral sign); the general antiderivative represents an infinity of functions as C can be any constant value.

Computing antiderivatives

Antiderivatives can be found by reversing the derivative rules.

The Sum Rule:

$$\int (f(x)+g(x))dx = \int f(x)dx + \int g(x)dx$$

The Difference Rule:

$$\int (f(x)-g(x))dx = \int f(x)dx - \int g(x)dx$$

The Constant Multiplier Rule:

$$\int (k\cdot f(x))dx = k\cdot \int f(x)dx$$

The (Integral) Power Rule:

$$\int x^n dx = \tfrac{1}{n+1}x^{n+1} + C$$

The (Integral) Exponential Function Rule:

$$\int b^x dx = \tfrac{1}{\ln b}b^x + C$$

The $\frac{1}{x}$ Rule:

$$\int \tfrac{1}{x} dx = \ln x + C$$