**Average value of a continuous function**

We know that a discrete function with inputs $x_1, x_2, \ldots, x_n$ has outputs $f(x_1), f(x_2), \ldots, f(x_n)$. The average value of the function is therefore the average of these $n$ numbers:

\[
\text{average value of the discrete function } f(x) = \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n}
\]

But if the function $f(x)$ is continuous, this method won’t work: there are infinitely many values of the function, so how can we add them up and divide by however many there are?!

We can estimate the average value of the continuous function $f(x)$ over the interval $a \leq x \leq b$ by means of an integral: if we dissect the interval $a \leq x \leq b$ into $n$ subintervals at the points

\[(a =) x_0, x_1, x_2, \ldots, x_n (= b)\]

so that each subinterval has equal width

\[
\Delta x = \frac{b - a}{n}
\]
then the right rectangle approximation method says that

\[
\int_a^b f(x) \, dx \approx f(x_1) \cdot \Delta x + f(x_2) \cdot \Delta x + \cdots + f(x_n) \cdot \Delta x \\
= \left( f(x_1) + f(x_2) + \cdots + f(x_n) \right) \cdot \Delta x \\
= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \cdot (\Delta x \cdot n) \\
= \frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \cdot (b - a)
\]

Therefore,

\[
\frac{f(x_1) + f(x_2) + \cdots + f(x_n)}{n} \approx \frac{\int_a^b f(x) \, dx}{b - a}.
\]

As we use more and more rectangles, the integral approximation by right rectangles improves in accuracy. But also, using more rectangles corresponds to averaging more and more values of the function on the interval \(a \leq x \leq b\), so that, in the limit as \(n \to \infty\), this discrete average approximates better and better the average value of the continuous function over the interval \(a \leq x \leq b\):
average value of the continuous function $f(x)$
\[
\frac{\int_a^b f(x)\,dx}{b - a}.
\]

If we let $\bar{y}$ represent this average value, we can write
\[
\bar{y} \cdot (b - a) = \int_a^b f(x)\,dx
\]
an equation which has a useful interpretation: on the right side, the integral measures the area under the curve $y = f(x)$ over the interval $a \leq x \leq b$, and the quantity on the left can be read as the area of a rectangle with base on the interval $a \leq x \leq b$ and height $\bar{y}$. In other words, $\bar{y}$ is the height of the rectangle on the interval $a \leq x \leq b$ whose area equals the area under the curve $y = f(x)$. 
We can use this idea of averaging a continuous function to return to an idea we investigated a while ago: the average value of the rate of change of the function $f(x)$ over the interval $a \leq x \leq b$ we know how to calculate as $\frac{f(b) - f(a)}{b - a}$. But if we do not have a way to compute the values of $f(a)$ and $f(b)$ directly, but instead have only information about the rate function $f'(x)$, we can still compute this average value by using the Fundamental Theorem. Since $\int_a^b f'(x)dx = f(b) - f(a)$, we obtain:

$$\text{average rate of change of } f(x) = \frac{\int_a^b f'(x)dx}{b - a}$$