From Archimedes' *Measurement of the Circle*.[1]

Proposition 1. *The area of any circle is equal to a right-angled triangle in which one of the sides about the right angle is equal to the radius, and the other to the circumference, of the circle.*[2]

Let $ABCD$ be the given circle, K the triangle described.

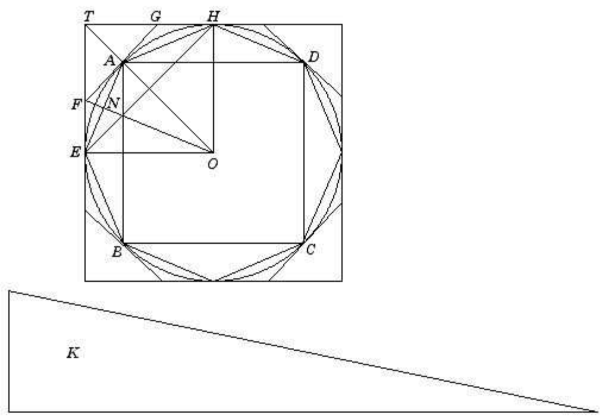

Figure 1: *Measurement of the Circle*, Prop. 1.

Then, if the circle is not equal to K, it must be either greater or less.[3]

If possible, let the circle be greater than K.[4]

[1]It has been argued convincingly that the shortness of this treatise by Archimedes – and the fact that the Greek of the text is not in the Doric dialect that Archimedes spoke and wrote – testifies that it is a copied fragment of a larger work of his on the quadrature of the circle.

[2]With this result, given the impact of *Elements*, II.14, Archimedes has shown that the quadrature of the circle can be effected provided it is possible to construct a straight line equal in length to the circumference of the circle. Of course, this just transferred the problem from a two-dimensional one to a one-dimensional one. It is just as difficult to **rectify** the curve – that is, straighten it – as it is to square the circle.

Most of us will remember from grade school that the circumference of a circle is given by the formula $C = 2\pi R$, where R is the radius. But since the area A of the circle satisfies $A = \pi R^2$, we can solve for π in the two formulas and equate the results to obtain

$$\frac{A}{R^2} = \frac{C}{2R},$$

or more simply, $A = \frac{1}{2}RC$, which we recognize as the area formula for a triangle with base C and height R, precisely equivalent to what Archimedes is claiming here.

Of course, this is an algebraic argument, not at all the way that Archimedes could have imagined the proof. He works only geometrically, by a masterful application of Eudoxus' method of exhaustion, as we will now see.

[3]Notice the resemblance of the structure of this argument to that of *Elements*, XII.2 (Chapter 6): a double *reductio ad absurdum*, with contradictions generated by inscribing and circumscribing polygons inside and outside the circle.

[4]The first half of the double *reductio* begins here.

Inscribe a square $ABCD$, bisect the arcs AB, BC, CD, DA, then bisect (if necessary) the halves, and so on, until the sides of the inscribed polygon whose angular points are the points of division subtend segments whose sum is less than the excess of the area of the circle over K.[5]

Thus the area of the polygon is greater than K.[6]

Let AE be any side of it, and ON the perpendicular on AE from the centre O.

Then ON is less than the radius of the circle and therefore less than one of the sides about the right angle in K. Also the perimeter of the polygon is less than the circumference of the circle, i.e., less than the other side about the right angle in K.

Therefore the area of the polygon is less than K;[7] which is inconsistent with the hypothesis.

Thus the area of the circle is not greater than K.[8]

If possible, let the circle be less than K.

Circumscribe a square, and let two adjacent sides, touching the circle in E, H, meet in T. Bisect the arcs between adjacent points of contact and draw the tangents at the points of bisection. Let A be the middle point of the arc EH, and FAG the tangent at A.[9]

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[5] Beginning with a square, Archimedes bisects the arcs between the corners of the polygon to double the number of division points around the circle and determine corner points of a (regular) polygon with double the number of sides, repeating the process until a sufficiently large enough portion of the circle has been exhausted by the polygon. How large is sufficiently large? Large enough that the remaining slivers of circular segments left uncovered by the polygon comprise an area smaller than the difference in area between the circle and K, as assumed at the outset.

[6] The inscribed polygon is always smaller than the circle, but it is so close in area to the circle that their difference in area is less than the difference in area between the circle and triangle K. Since the circle is also larger than K, by assumption, the area of the polygon is something between that of the circle and K.

[7] Archimedes recognizes that the n-sided polygon can be decomposed into n equal triangles by radial lines emanating from the center of the circle. Each of these triangles has a height equal to ON, and base equal to one of the sides of the polygon. One can then imagine another series of n triangles (Figure 2), all of whose bases are equal to the side of the n-sided polygon, set side by side along a line and again sharing their opposite vertex so that they all have height equal to ON. This collection of triangles has total area equal to that of the polygon. But since this composite triangle has a base equal to the total perimeter of the polygon, which is less than the circumference of the circle, and height ON less than the radius of the circle, its area is less than that of K.

[8] The first half of the double *reductio* argument is thus complete.

[9] The technique here in the second half of the proof is entirely similar to that of the first

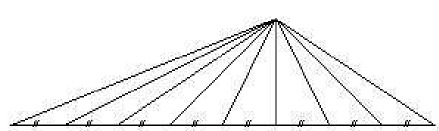

Figure 2: A triangle equal to the polygon.

Then the angle TAG is a right angle.[10]

Therefore $TG > GA = GH$.[11] It follows that the triangle FTG is greater than half the area $TEAH$.[12]

Similarly, if the arc AH be bisected and the tangent at the point of bisection be drawn, it will cut off from the area GAH more than one-half.[13]

Thus, by continuing the process, we shall ultimately arrive at a circumscribed polygon such that the spaces intercepted between it and the circle are together less than the excess of K over the area of the circle.

Thus the area of the polygon will be less than K.[14]

Now, since the perpendicular from O on any side of the polygon is equal to the radius of the circle, while the perimeter of the polygon is greater than the circumference of the circle, it follows that the area of the polygon is greater than the triangle K;[15] which is impossible.

Therefore the area of the circle is not less than K.[16]

Since then the area of the circle is neither greater nor less than K, it is equal to it.

Proposition 2. *The area of a circle is to the square on its diameter as 11 to 14.*[17]

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half. Starting with a square, a succession of polygons is considered, each circumscribing the circle – guaranteeing that all contain, hence are larger than, the circle – and each having twice as many sides as the previously considered polygon, so as to more closely equal the circle.

[10]Recall *Elements*, III.18 from Chapter 7.

[11]$TG > GA$ because the hypotenuse of right triangle TAG is longer than either of its sides.

[12]By area $TEAH$, Archimedes means that portion of the triangle TEH which lies outside the circle.

[13]The need to ensure that more than half of these triangular areas are used up is vital for the success of the method of exhaustion. Recall that in the middle of the proof of *Elements*, XII.2 in Chapter 6 a crucial reference to Proposition X.1 is made; this proposition states that if more than half of any area is removed, again and again in succession, eventually the area remaining will be smaller than any preassigned amount; that is, it will eventually become exhausted. This is Archimedes' goal: to ensure that this successive doubling of the sides of the circumscribed polygon will eventually result in one so close in area to that of the circle that the difference between it and the circle is less than the difference between K and the circle.

[14]Since both the circumscribed polygon and K are larger than the circle, but the difference between the polygon and the circle is less than the difference between K and the circle, then K must be larger than the polygon.

[15]This is the mirror conclusion to that in the first part of the double *reductio* argument: by decomposing the polygon into n equal triangles by lines radiating from O to its corners, we can see that the polygon is equal in area to a triangle composed of n triangles with bases each equal to a side of the polygon and height equal to the radius of the circle, as in Figure 2. But the composite triangle will have a base equal to the perimeter of the polygon, which is greater than the circumference of the circle and height equal to the radius of the circle, so its area must be less than that of K.

[16]This nicely completes the second part of the double *reductio*.

[17]Recalling that the area A of a circle with radius R and diameter $D = 2R$ is given by the formula

$$A = \pi R^2 = \pi \left(\frac{D}{2}\right)^2 = \frac{\pi}{4} D^2,$$

we know that the ratio of the area of a circle to the square on its diameter is exactly $\frac{\pi}{4}$. If Archimedes were to be taken literally here, we would be able to claim that $\frac{\pi}{4} = \frac{11}{14}$ which

Proposition 3. *The ratio of the circumference of any circle to its diameter is less than $3\frac{1}{7}$ but greater than $3\frac{10}{71}$.*[18]

Let AB be the diameter of any circle, O its center, AC the tangent at A; and let angle AOC be one-third of a right angle.[19]

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implies that $\pi = \frac{22}{7}$. While there is close agreement between these numbers – and the use of the fraction $\frac{22}{7}$ is a common and very good approximation to π, it is not *exactly* true. As Heath notes in his 1897 English edition of this work of Archimedes [**?**], "The text of this proposition is not satisfactory." Clearly, if Archimedes did indeed state this proposition (it could have been inserted by a later editor), he must have meant it in the sense of an approximation to the value of the ratio of circle and square on diameter.

In addition, Heath goes on to say that "Archimedes cannot have placed it before Proposition 3, as the approximation depends on the result of that proposition." Indeed, it does follow as a corollary directly from Proposition 3 (see the Exercises at the end of this chapter), so no separate proof was deemed necessary.

[18] We will quote Heath again here, who finds the editorial comments of the sixth century CE commentator, Eutocius of Askalon, quite useful:

> In view of the interesting questions arising out of the arithmetical content of this proposition of Archimedes, it is necessary, in reproducing it, to distinguish carefully the actual steps set out in the text as we have it from the intermediate steps (mostly supplied by Eutocius) which it is convenient to put in for the purpose of making the proof easier to follow. Accordingly all the steps not actually appearing in the text have been enclosed in square brackets, in order that it may be clearly seen how far Archimedes omits actual calculations and only gives results.

We, too, will retain most of the editorial comments that Eutocius/Heath supplied, as Archimedes is difficult to follow here for a reader not steeped in the milieu of third century Greek geometry. He only shows the finished product of his computations, or rather, the editions of his work that survive for us have this characteristic; it may be that his original manuscripts, now lost to us, were more forthcoming with the details. The reader must work hard to fill in the gaps left in the proof to understand how he was able to make the next claim, and luckily Eutocius (and Heath as well) have done this for us. In some cases, it is impossible to reconstruct his steps absolutely because multiple paths can be offered to arrive at the same conclusions.

The proof is structured in two parts, first to show that the ratio of circumference to diameter of a circle – and we will continue to call this ratio π despite the fact that Archimedes would not have done so – is less than $\frac{22}{7}$ by estimating measurements of the perimeter of circumscribed polygons to the circle; and then to show that the ratio is more than $\frac{223}{71}$ by estimating measurements of the perimeter of inscribed polygons.

[19] Archimedes begins by considering the geometry of the circumscribed regular hexagon. In the diagram (Figure 3), C is one of the corners of this hexagon and A is the point of tangency of one of the sides having C as endpoint, hence it is also the midpoint of that side of the polygon. The opposite corner of this side is beyond the point H and does not appear in the diagram; let us call it C'. Then angle $C'OC$ is the central angle of a regular hexagon, and so it measures $(360° \div 6 =)\ 60°$, whence angle AOC is $30°$, or as Archimedes says "one-third of a right angle." But as angle ACO is also $60°$, we see that triangle AOC must be a $30°$-$60°$-$90°$ triangle.

Therefore, a simple application of the Pythagorean theorem shows (see the Exercises) that the sides of this triangle are in the proportions $OC : OA : AC = 2 : \sqrt{3} : 1$. This is Archimedes' next assertion, except that he does this in the form of an inequality by replacing $\sqrt{3}$ with a very good rational approximation, the slighty-too-large ratio $265 : 153$. How did Archimedes come up with this approximation? While no one can say for sure, but the best explanation seems to be that he used a number of anthyphairetic steps (see the discussion in Chapter 4 on *anthyphairesis*) with the segments OA and AC to find this good approximation. (One of this chapter's exercises will duplicate this computation.)

Once the ratio $OA : AC$ is computed, the ratio $OC : AC$, known to be equal to $2 : 1$, is

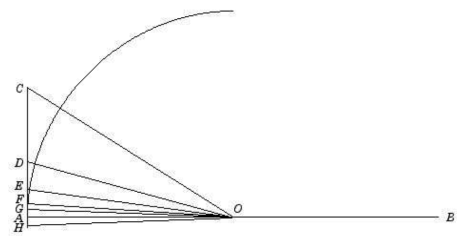

Figure 3: *Measurement of the Circle*, Prop. 3: a portion of the circumscribed polygons.

Then[20]

$$OA : AC\,[= \sqrt{3} : 1] > 265 : 153 \tag{1}$$

and

$$OC : CA\,[= 2 : 1] = 306 : 153. \tag{2}$$

First, draw OD bisecting the angle AOC and meeting AC in D.[21] Now [by *Elements* VI.3[22]]

$$CO : OA = CD : DA$$

easy to express in similar terms.

[20]Beginning here and throughout the rest of this text are displayed a number of equations and inequalities involving the ratios with which Archimedes is working. It is necessary to emphasize that this is is to make Archimedes' proof intelligible to a modern reader without undue effort and not because Archimedes would have written his mathematics in this way. The use of signs like $+$, $-$, and even $=$ and $<$, is extremely useful for modern readers, but none of these symbols was available to Archimedes. Instead, he would have written out in words the content of all these relations, making the language very difficult to parse in this form.

[21]The point D will be one corner of the regular dodecagon (having 12 sides) that circumscribes the circle; the opposite corner of this side of the dodecagon whose midpoint is A is also beyond the point H and does not appear in the diagram; let us call it D'. Then angle $D'OD$ is a central angle of this polygon, and so it measures $(360° \div 12 =)\ 30°$, whence angle AOD is $15°$, or half of angle AOC.

[22]This proposition states that the angle bisector of any angle in a triangle cuts the opposite side into two parts that have a ratio equal to the ratio of the sides of the original triangle which are on the corresponding sides of the bisector. In the context of the triangle AOC, OD bisects angle AOC, so the proposition allows to Archimedes to make this next assertion. (A proof of *Elements* VI.3 is in the Exercises.)

so that[23]

$$[CO + OA : OA = CA : DA$$

or]

$$CO + OA : CA = OA : DA.$$

Therefore [by adding (1) and (2)]

$$OA : AD > 571 : 153\,. \tag{3}$$

Hence[24]

$$\begin{array}{rll} OD^2 : AD^2 & [= & \left(OA^2 + AD^2\right) : AD^2 \\ & > & (571^2 + 153^2) : 153^2\ ] \\ & > & 349450 : 23409 \end{array}$$

so that[25]

$$OD : DA > 591\tfrac{1}{8} : 153\,. \tag{4}$$

Secondly,[26] let OE bisect the angle AOD, meeting AD in E. [Then

$$DO : OA = DE : EA$$

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[23]Expressed as an equation, the previous proportion is

$$\frac{CO}{OA} = \frac{CD}{DA}.$$

Adding one to both sides, we obtain

$$\frac{CO}{OA} + \frac{OA}{OA} = \frac{CD}{DA} + \frac{DA}{DA}$$

which on combining the fractions yields

$$\frac{CO + OA}{OA} + \frac{OA}{OA} = \frac{CD + DA}{DA} + \frac{DA}{DA}.$$

[24]This next computation is an application of the Pythagorean Theorem to triangle AOD.

[25]Archimedes will now evaluate a square root. The number 23409 is a perfect square so that causes no difficulty, but 349450 is not a perfect square: he needs to find a useful approximation. He can work out that the square root of 349450 is very close to $591\frac{1}{7}$ and so he settles on $591\frac{1}{8}$ as a convenient lower bound.

[26]Before we move on, compare lines (1) and (2) with (3) and (4). The first set gives estimates of measures of the sides of triangle AOC, a portion of the circumscribed hexagon. The second set gives estimates of corresponding measures for triangle AOD, a portion of the circumscribed dodecagon. In both sets of relations we have ratios between OA, the radius of the circle, and either the half-length of the side of the circumsribed polygon (AC in line (1) and AD in line (3)), or the distance from the center to a corner of the polygon (OC in line (2) and OD in line (4)).

Archimedes has completed the first stage of the computational analysis. From line (1), he could determine the ratio of the perimeter of the hexagon (which is 12 times the length of AC) to the diameter of the circle (twice the radius OA); this would give a crude overestimate to the ratio of the circumference of the circle to its diameter. In order to achieve better accuracy, Archimedes leaves aside the hexagon and concentrates instead on measuring the dodecagon, whose perimeter is closer in length to that of the circle's circumference. But since lines (3) and (4) bring him to the same point with the dodecagon as did lines (3) and (4) for the hexagon, he sees that he can carry this same iterative procedure forward to obtain measures for the corresponding line segments in the circumscribed 24-gon obtained by bisecting the central so that

$$DO + OA : DA = OA : AE\,.]$$

Therefore

$$\begin{aligned} OA : AE \quad & [> (591\tfrac{1}{8} + 571) : 153 \quad \text{by (3) and (4) }] \\ & > 1162\tfrac{1}{8} : 153 \end{aligned} \tag{5}$$

[It follows that

$$\begin{aligned} OE^2 : EA^2 \quad & > (\{1162\tfrac{1}{8}\}^2 + 153^2) : 153^2 \\ & > (1350534\tfrac{33}{64} + 23409) : 23409 \\ & > 1373943\tfrac{33}{64} : 23409\,] \end{aligned}$$

Thus[27]

$$OE : EA > 1172\tfrac{1}{8} : 153\,. \tag{6}$$

Thirdly,[28] let OF bisect the angle AOE and meet AE in F. We thus obtain the result [corresponding to (3) and (5) above] that

$$\begin{aligned} OA : AF \quad & [> (1162\tfrac{1}{8} + 1172\tfrac{1}{8}) : 153\,] \\ & > 2334\tfrac{1}{4} : 153 \end{aligned} \tag{7}$$

[Therefore

$$\begin{aligned} OF^2 : FA^2 \quad & > (\{2334\tfrac{1}{4}\}^2 + 153^2) : 153^2 \\ & > 5472132\tfrac{1}{16} : 23409\,] \end{aligned}$$

Thus[29]

$$OF : FA > 2339\tfrac{1}{4} : 153 \tag{8}$$

Fourthly,[30] let OG bisect the angle AOF, meeting AF in G. We have then

$$\begin{aligned} OA : AG \quad & [> (2334\tfrac{1}{4} + 2339\tfrac{1}{4}) : 153 \quad \text{by means of (7) and (8) }] \\ & > 4673\tfrac{1}{2} : 153 \end{aligned}$$

Now the angle AOC, which is one-third of a right angle, has been bisected four times, and it follows that

$$\angle AOG = \tfrac{1}{48}(\text{a right angle}).$$

angles of the dodecagon.

This is precisely what he does. Indeed, he will not stop there, continuing on to obtain measures for the corresponding line segments in the circumscribed 48-gon, and then again for the circumscribed 96-gon, where he finally pauses to give his final highly accurate estimate.

[27] Once more Archimedes requires the computation of a convenient lower bound for the square root of a number, $1373943\frac{33}{64}$, which is clearly not a perfect square.

[28] Lines 5 and 6 are for the 24-gon what lines 3 and 4 are for the dodecagon. Now Archimedes moves on to step four of this five step iteration, the measurement of the circumscribed 48-gon.

[29] This is the last of the square root estimates.

[30] This is the fifth and final step of the iteration, to measure the half-side of the circumscribed 96-gon.

Make the angle AOH on the other side of OA equal to the angle AOG, and let GA produced meet OH in H.

Then

$$\angle GOH = \tfrac{1}{24}(\text{a right angle}).$$

Thus GH is one side of a regular polygon of 96 sides circumscribed to the given circle.

And, since $OA : AG > 4673\frac{1}{2} : 153$ while $AB = 2OA$, $GH = 2AG$, it follows that

$$\begin{aligned} AB : (\text{perimeter of polygon of 96 sides}) \quad & [> \quad 4673\tfrac{1}{2} : 153 \times 96\,] \\ & > \quad 4673\tfrac{1}{2} : 14688\,. \end{aligned}$$

But

$$\frac{14688}{4673\frac{1}{2}} = 3 + \frac{667\frac{1}{2}}{4673\frac{1}{2}} \left[< 3 + \frac{667\frac{1}{2}}{4672\frac{1}{2}} \right] < 3\tfrac{1}{7}\,.$$

Therefore the circumference of the circle (being less than the perimeter of the polygon) is *a fortiori* less than $3\frac{1}{7}$ times the diameter AB.[31]

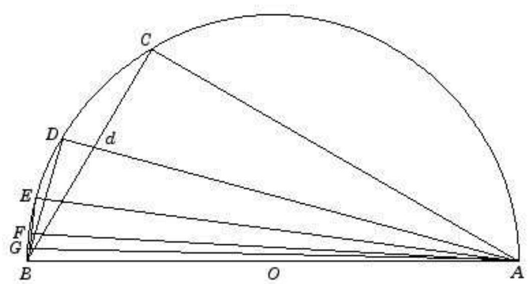

Figure 4: *Measurement of the Circle*, Prop. 3: working with inscribed polygons.

Next let AB be the diameter of a circle, and let AC, meeting the circle in C, make the angle CAB equal to one-third of a right angle. Join BC.[32] Then[33]

$$AC : CB[= \sqrt{3} : 1\,] < 1352 : 780\,.$$

[31]Notice that in the last line of computation above, Archimedes has inverted the ratio he has been working with, which forces the inequalities to invert as well:

$$(\text{circumference}) : AB < (\text{perimeter of 96-gon}) : AB < 14688 : 4673\tfrac{1}{2} < 3\tfrac{1}{7}\,.$$

This ends the first part of the analysis. Now, Archimedes will consider making estimates of ratios of lines having to do with the inscribed regular hexagon, dodecagon, 24-gon, 48-gon, and 96-gon. These will allow him to give an accurate underestimate for the ratio of circumference to diameter of the circle.

[32]Recall from note **??** that according to *Elements*, III.20, angle CAB is half as large as the central angle COB (not drawn in the figure). So if angle CAB is 30° (one-third of a right angle), then angle COB is 60°. Therefore BC is one side of an inscribed regular hexagon.

[33]Recall (Exercise 7.4) that the side of the inscribed hexagon is equal to the radius of the circle, so $AB : BC = 2 : 1$. Also, by *Elements* III.20, angle ACB is right. So by the Pythagorean Theorem, $AC : CB = \sqrt{3} : 1$. Once again, therefore, Archimedes needs a convenient approxi-

First,[34] let AD bisect the angle BAC and meet BC in d and the circle in D. Join BD. Then[35]

$$\angle BAD = \angle dAC = \angle dBD$$

and the angles at D, C, are both right angles.

It follows that the triangles ADB, $[ACd,]$ BDd are similar.

Therefore[36]

$$\begin{aligned} AD : DB &= BD : Dd \\ [&= AC : Cd\,] \\ &= AB : Bd \quad \text{[by \textit{Elements}, VI.3]} \\ &= (AB + AC) : (Bd + Cd) \end{aligned}$$

or

$$BA + AC : BC = AD : DB. \tag{9}$$

[But $AC : CB < 1351 : 780$ from above, while

$$BA : BC = 2 : 1 = 1560 : 780.\ ]$$

Therefore

$$AD : DB < 2911 : 780 \tag{10}$$

[Hence

$$\begin{aligned} AB^2 : BD^2 &< (2911^2 + 780^2) : 780^2 \\ &= 9082321 : 608400\ ] \end{aligned}$$

Thus[37]

$$AB : BD < 3013\tfrac{3}{4} : 780 \tag{11}$$

Secondly,[38] let AE bisect the angle BAD, meeting the circle in E; and let BE be joined.

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mation – this time an underestimate – to $\sqrt{3}$. By extending the anthyphairetic procedure that allowed him to get the estimate in line (1) above, he arrives at the approximation 1351 : 780. (See the chapter Exercises for details.)

[34] The computations of the ratios of the sides of triangle ABC give Archimedes the first set of measurements for this part of the analysis. Here BC is the side of the inscribed hexagon; in successive stages of the iteration, he will compute estimates for the corresponding ratios in the inscribed dodecagon, 24-gon, 48-gon, and 96-gon. Each of these is obtained simply by bisecting the angle of the triangle at A.

[35] Since angle CAB measures 30°, angle DAB measures 15°, so angle DOB (not drawn) is 30° and BD is the side of the inscribed dodecagon. Triangles dAC and dBD have equal angles at d and both have right angles: one at C, the other at D. Therefore, the third angle in each triangle has the same measure: angle dAC = angle dBD.

[36] The first two lines below rely on the similarity of the triangles just mentioned. The third line has a citation for *Elements*, VI.3. See note 22 above. The proposition is being applied to triangle ABC with Ad as angle bisector. Finally, the last step is equivalent to the arithmetical fact that if $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} = \frac{a+c}{b+d} = \frac{c}{d}$, whose verification is left to the chapter Exercises.

[37] Lines (10) and (11) give the estimated measures for the sides of triangle ABD, which apply to the inscribed dodecagon.

[38] Now, in step three of the iterated computations, Archimedes concentrates on measuring the inscribed regular 24-gon.

Then we prove, in the same way as before,[39] that

$$\begin{aligned} AE : EB\ [&= BA + AD : BD \\ &< (3013\tfrac{3}{4} + 2911) : 780 \quad \text{by (9) and (10)]} \\ &= 5924\tfrac{3}{4} : 780 \\ &= 5924\tfrac{3}{4} \times \tfrac{4}{13} : 780 \times \tfrac{4}{13} \\ &= 1823 : 240\ . \end{aligned} \tag{12}$$

[Hence

$$\begin{aligned} AB^2 : BE^2 &< (1823^2 + 240^2) : 240^2 \\ &= 3380929 : 57600\ .\ ] \end{aligned}$$

Therefore

$$AB : BE < 1838\tfrac{9}{11} : 240 \tag{13}$$

Thirdly,[40] let AF bisect the angle BAE, meeting the circle in F. Thus

$$\begin{aligned} AF : FB\ [&= (BA + AE) : BE \\ &< 3661\tfrac{9}{11} : 240 \quad \text{by (12) and (13)]} \\ &= 3661\tfrac{9}{11} \times \tfrac{11}{40} : 240 \times \tfrac{11}{40} \\ &= 1007 : 66 \end{aligned} \tag{14}$$

[It follows that

$$\begin{aligned} AB^2 : BF^2 &< (1007^2 + 66^2) : 66^2 \\ &= 1018405 : 4356 \end{aligned}$$

Therefore

$$AB : BF < 1009\tfrac{1}{6} : 66 \tag{15}$$

Fourthly,[41] let the angle BAF be bisected by AG, meeting the circle in G. Then

$$AG : GB\ [= (BA + AF) : BF\ ] < 2016\tfrac{1}{6} : 66 \quad \text{[by (14) and (15).}$$

And

$$\begin{aligned} AB^2 : BG^2 &< (\{2016\tfrac{1}{6}\}^2 + 66^2) : 66^2 \\ &= 4069284\tfrac{1}{36} : 4356\ .\ ] \end{aligned}$$

Therefore $AB : BG < 2017\frac{1}{4} : 66$ whence

$$BG : AB > 66 : 2017\tfrac{1}{4} \tag{16}$$

[39] The proportion in the first line of the next computation follows in exactly the same way for the 24-gon in the same way that line (9) was derived for the dodecagon.

[40] Next come the estimates for the ratio measures corresponding to the 48-gon.

[41] And the final step obtains estimates for the ratio measures corresponding to the 96-gon.

[Now the angle BAG which is the result of the fourth bisection of the angle BAC, or of one-third of a right angle, is equal to one-fortyeighth of a right angle.

Thus the angle subtended by BG at the centre[42] is $\frac{1}{24}$(a right angle).

Therefore BG is a side of a regular inscribed polygon of 96 sides.

It follows [from (16)] that

$$\begin{aligned} \text{(perimeter of 96-gon)} : AB\ [&= 96 \times 66 : 2017\tfrac{1}{4}\ ] \\ &= 6336 : 2017\tfrac{1}{4}\,. \end{aligned}$$

And

$$\frac{6336}{2017\frac{1}{4}} > 3\tfrac{10}{71}$$

.

Much more then is the circumference of the circle greater than $3\frac{10}{71}$ times the diameter.

Thus the ratio of the circumference to the diameter $< 3\frac{1}{7}$ but $> 3\frac{10}{71}$.

[42] This is another application of *Elements* III.20 (see note (**??**)).