The Axiomatic Method in Mathematics

The standard methodology for modern mathematics has its roots in Euclid’s (3rd c. BCE) organization of geometry and arithmetic in his famous *Elements*. Geometers in the eighteenth and nineteenth centuries formalized this process even more, and their successes in geometry were extended throughout all of mathematics, so that today, the axiomatic method pervades every mathematical theory.

This **axiomatic method** is employed to give reliable and objective reasons for why conjectures about mathematical objects hold true. It is based on logically deductive argumentation that establishes **proofs** for every assertion in the **theory** under investigation. These assertions are called **propositions**, or **theorems** (when their statements are important milestones in the organization of the theory), or **lemmas** (when their purpose is primarily to establish an important step in the proof of a future theorem), or **corollaries** (when they are immediate consequences of a theorem).

Proofs of theorems are arguments based on the truth of previously proven theorems, but as these previous proofs require the establishment of prior assertions for their truth, we have to avoid an
infinite regress of proof: the theory must be founded on assertions that do not require proof. Such statements are called the axioms of the theory.

Axioms make assertions about mathematical objects that are central to the theory, and the terms used in the axioms require careful definition so that the deductive method suffers no ambiguities about what the theory claims is true. But as with the proofs of theorems, the definitions of these terms involve terms that must themselves be defined carefully, so we become involved in an infinite regress of definitions! The way to avoid this is to begin with a list of undefined terms upon which all terms in the theory are defined. The undefined terms therefore have meanings that are left up to interpretation.

In geometry, the undefined terms are point, line, plane, and the fundamental relation among these objects of “lies on.” The standard interpretation of these terms is the familiar one that conceives of points as locations in space, lines as straight, infinitely long, one-dimensional curves in space consisting of an infinite collection of points, and planes as flat surfaces of infinite extent made up of all the points located on that surface. The relation “lies on” is interpreted as containment: a point lies on a line or plane if it is one of the points of that line or plane, and a line lies on a plane when all the
points of the line are points of the plane. As such, however, this is only one possible interpretation of these undefined terms (even though it is the most important). We will say more about this below.

Once the undefined terms of an axiomatic theory are laid down, all other terms are defined relative to these. Axioms are then posed regarding these objects to found the theory. The axioms are chosen for
• **convenience**: they lead to an interesting theory;
• **efficiency**: the important theorems in the theory can be derived in a short number of steps; and
• **plausibility**: the statements of the axioms are compelling and believeable, often selected on the basis of experience or experimentation.

Furthermore, a set of axioms is valuable for its
• **consistency**: it is impossible to derive contradictory statements from them;
• **independence**: none of the axioms is logically redundant, that is, it cannot be deduced from the other axioms (for then it can be removed from the list of axioms to become a theorem instead); and
• **completeness**: every true statement about the objects of the theory is derivable from the axioms.
Any environment in which the undefined terms have an interpretation and the axioms hold is called a **model** for that axiomatic system. For instance, the standard interpretations of point, line, plane and "lies on" that were described above provides a model for Euclidean geometry. All theorems derivable from the axioms of the system are valid in any model for the system.

Two models $\mu$ and $\mu'$ are **isomorphic** if there is a one-to-one correspondence between the objects of the two models so that if $x$ and $y$ are two undefined terms in model $\mu$ and some relationship holds in the theory between $x$ and $y$, then the same relationship of the theory will hold between the corresponding objects $x'$ and $y'$ in $\mu'$. (See Example 4, p. 65, where three nonisomorphic models are given for the same axiomatic system.)

Some axiomatic systems have only one model up to isomorphism. that is, any two models $\mu$ and $\mu'$ for the system must be isomorphic to each other. We call such systems **categorical**. It was an important discovery by David Hilbert in 1860s that Euclidean geometry (as he axiomatized it) is a categorical axiomatic system.