Exam 1 Practice

Problem 1. (3 pts.) What is the relationship, if any, between Modus Tollens and Modus Ponens?

Modus Tollens is simply Modus Ponens on the contrapositive of the original statement. Since a statement and its contrapositive are equivalent, if Modus Ponens is a valid argument form, then so is Modus Tollens.

Problem 2. (5 pts.) In Samuel Clemens’ old house you discover a note in the cupboard about a hidden manuscript. The note contains five true statements about the manuscript’s location; where is the manuscript? Don’t forget to show your work as to how you determine the manuscript’s location.

1. If this house is next to a lake, the manuscript is not in the kitchen.
2. If the tree in the front yard is an elm, then the manuscript is in the kitchen.
3. This house is next to a lake.
4. The tree in the front yard is an elm or the manuscript is buried under the flagpole.
5. If the tree in the back yard is an oak, then the manuscript is in the garage.

The manuscript is buried under the flagpole.

Problem 3. (8 pts.) You know what to do: deduce the conclusion from the premises giving the reason for each step.

\[
\begin{align*}
\sim p \lor q & \rightarrow r \\
s \lor \sim q & \\
\sim t & \\
p & \rightarrow t \\
\sim p \land r & \rightarrow \sim s \\
\therefore \sim q
\end{align*}
\]

\[
\begin{align*}
\sim p & \quad \text{M.T w/3 and 4} \\
\sim p \lor q & \quad \text{Generalization w/6} \\
r & \quad \text{M.P w/1 and 7} \\
\sim p \land r & \quad \text{Conjunction w/6 and 8} \\
\sim s & \quad \text{M.P. with 5 and 9} \\
\therefore \sim q & \quad \text{Elimination w/2 and 10}
\end{align*}
\]
Problem 4. (5 pts.) What is the difference between propositional logic (or calculus) and predicate logic (or calculus)?

Propositional logic deals with statements - a sentence that must be either true or false (but not both). Predicates are sentence fragments that are turned into sentences (and hence statements) by instantiating the predicate with one or a set of values. The Universal and Existential quantifiers are two such ways to instantiate this set.

Problem 5. (3 pts.) What is a statement? What is a predicate? What is a truth set? How, if at all, are these three terms related?

A statement is a sentence that must be either true or false, but not both. Predicates are sentence fragments that are turned into sentences (and hence statements) by instantiating the predicate with one or a set of values. A predicate’s truth set are those values for which the predicate becomes a true statement.

Problem 6. (4 pts.) Rewrite the following statement, and its negation, using quantifiers and variables:

Somebody loves everybody

\[ \exists x \in P \text{ such that } \forall y \in P, Loves(x, y) \]
\[ \forall x \in P, \exists y \in P \text{ such that } \sim Loves(x, y) \]

where Loves(x, y) is to mean that x loves y. The negation is to mean everyone has someone they do not love.

Problem 7. (3 pts.) Write the negation of the following statement:

\[ \forall x, z \in D, \exists y \in E \text{ such that } (P(x) \lor \sim Q(y)) \iff R(z) \]

\[ \exists x, z \in D \text{ such that } \forall y \in E((P(x) \lor \sim Q(y)) \land \sim R(z)) \lor (R(z) \land (\sim P(x) \land Q(y))) \]

Problem 8. (4 pts.) Rewrite the following statement in both English without using the word “necessary,” and using formal notation:

Being on time each day is a necessary condition for keeping this job.

If a person is not on time each day, then that person will not keep this job.

\[ \forall x \in P, \sim OnTime(x) \rightarrow \sim JobKeep(x) \equiv \forall x \in P, JobKeep(x) \rightarrow OnTime(x) \]
Problem 9. (7 pts.) Prove that the square of any integer has the form $4k$ or $4k+1$ for some integer $k$.

To Prove: $\forall n \in \mathbb{Z}, n^2 = 4k + 1$ or $n^2 = 4k$ for some integer $k$

Proof: Let $n$ be any integer. By the quotient remainder theorem, $n = 4q$ or $4q + 1$, or $4q + 2$, or $4q + 3$ for some integer $q$.

Case 1 ($n = 4q$): $n^2 = (4q)^2 = 16q^2 = 4(4q^2)$.

Case 2 ($n = 4q + 1$): $n^2 = (4q + 1)^2 = 16q^2 + 8q + 1 = 4(4q^2 + 2q) + 1$

Case 3 ($n = 4q + 2$): $n^2 = (4q + 2)^2 = 16q^2 + 16q + 4 = 4(4q^2 + 4q + 1)$

Case 4 ($n = 4q + 3$): $n^2 = (4q + 3)^2 = 16q^2 + 24q + 9 = 16q^2 + 24q + 8 + 1 = 4(4q^2 + 6q + 2) + 1$

Problem 10. (8 pts.) Describe the four methodologies studied for proving (or disproving) the truthiosity of a quantified statement.

Assume that the statement to be proved is $\forall x \in D, P(x) \rightarrow Q(x)$

- Direct proof. Starting with $P(x)$ and the technique of generalizing from the generic particular, one derives the statement of $Q(x)$.

- Counterexample. Demonstrate at least one case if $x \in D$ where $P(x)$ is true but $Q(x)$ is false.

- Indirect proof: Contradiction. Assert that the negation is true: $\exists x \in D$ such that $P(x) \land \sim Q(x)$ and that you have such an $x$ that satisfies $Q(x)$. Show that this leads (using the methodology of direct proof) to a contradiction. i.e. Derive $\sim P(x)$.

- Indirect proof: Contraposition: Prove $\forall x \in D, \sim Q(x) \rightarrow P(x)$. Again, one uses the techniques of direct proof. Begin, using a generalization from the generic particular, with $\sim Q(x)$ and derive $\sim P(x)$. Hopefully, one sees the similarity between this proof technique and the proof by contradiction: Start with $\sim Q(x)$ and derive $\sim P(x)$.

Problem 11. (5 pts.) For all integers $m$ and $n$, if $m + n$ is even then $m$ and $n$ are both even or both odd.

To Prove: $\forall n, m \in \mathbb{Z}, \text{Even}(n + m) \rightarrow \text{BothEven}(n, m) \lor \text{BothOdd}(n, n)$

Proof: Proof by Contradiction: Assume

$\exists n, m \in \mathbb{Z}$ such that $\text{Even}(n + m) \land \sim \text{BothEven}(n, m) \land \sim \text{BothOdd}(n, m)$
Since $n$ and $m$ are not both even nor both odd, one must be even and one must be odd. Let $n$ be the even value. $n = 2k$ and $m = 2j + 1$. $n + m = 2k + 2j + 1 = 2(j + k) + 1$ which is odd. Since $n + m$ cannot be both even and odd we have a contradiction, hence since the negation of the original statement is false, the original statement is true.

**Problem 12.** (5 pts.) Prove that for any odd integer

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}$$

**To Prove:** $\forall x \in \mathbb{Z}, \text{Odd}(x) \rightarrow \left\lfloor \frac{n^2}{4} \right\rfloor = \frac{n^2 + 3}{4}$

**Proof:** If $x$ is odd, then $x = 2k + 1$. Hence:

$$\left\lfloor \frac{n^2}{4} \right\rfloor = \left\lfloor \frac{(2k + 1)^2}{4} \right\rfloor = \left\lfloor \frac{4k^2 + 4k + 1}{4} \right\rfloor = \left\lfloor k^2 + k + \frac{1}{4} \right\rfloor = k^2 + k + 1$$

Now substituting $n = 2k + 1$ into $\frac{n^2 + 3}{4} = \frac{(2k+1)^2+3}{4}$

$$\frac{(2k + 1)^2 + 3}{4} = \frac{4k^2 + 4k + 1 + 3}{4} = \frac{4k^2 + 4k + 1}{4} = k^2 + k + 1$$

By showing that both sides equal $k^2 + k + 1$ we have shown the equality and hence the theorem. $\blacksquare$