Function Transformations

Now that we have carefully investigated four basic types of commonly occurring functions (linear, quadratic, exponential, and logarithmic), we will widen our view to encompass a larger collection of functions. In particular, we will pay special attention to ways in which we can transform one function into another by certain simple steps, considering what effects these transformations have on the graphs and formulas of the functions we work with.

The first and simplest of these transformations corresponds to shifting the graph of a function vertically or horizontally in a uniform manner. A \textbf{vertical shift transformation} moves the entire graph of the function up a fixed number of units (a downward shift can be realized as moving the graph up by a \textit{negative} amount). This has the effect of adding some constant value (which may be negative) to each of the outputs of the given function to obtain the transformed function. Symbolically, if \( y = f(x) \) is the given function and \( k \) the transformation constant, then the transformed function has the form \( y = g(x) = f(x) + k \); the graph of \( g \) is the same as that of \( f \) but shifted \( k \) units up.

[\textit{Examples}: p. 188, #1(c), 5]
A **horizontal shift transformation** moves the entire graph of the function to the left a fixed number of units (a shift to the right can be realized by moving the graph to the left by a *negative* amount). This has the effect of adding some constant value (which may be negative) to the *outputs* of the given function in order to obtain the transformed function. Symbolically, if \( y = f(x) \) is the given function and \( k \) the transformation constant, then the transformed function has the form \( y = g(x) = f(x + k) \) (note the subtle but very important difference between this transformation and the earlier one!); the graph of \( g \) is the same as that of \( f \) but shifted \( k \) units to the left.

[Examples: p. 188, #1(a,b), 6, 37]

Another important transformation arises from reflection of the graph of a function across the \( x \)- or \( y \)-axis. **Reflection across the \( x \)-axis** replaces each point \( (x, y) \) on the graph with the symmetric point \( (x, -y) \). Symbolically, if \( y = f(x) \) is the given function, then the transformed function has the form \( y = g(x) = -f(x) \).

[Example: p. 197, #5]

**Reflection across the \( y \)-axis** replaces each point \( (x, y) \) on the graph with the symmetric point \( (-x, y) \). Symbolically, if \( y = f(x) \) is the given function, then
the transformed function has the form
\[ y = g(x) = f(-x) \] (again, notice the subtle difference between this and the previous situation!).

[Example: p. 197, #4]

An interesting situation arises when we use both types of reflection to transform the function: now, point \((x, y)\) on the graph is replaced with the symmetric point \((-x, -y)\). This corresponds to symmetry with respect to the origin. Symbolically, if \(y = f(x)\) is the given function, then the transformed function has the form \(y = g(x) = -f(-x)\).

[Example: p. 197, #6, 23]

Some important functions have the property that when they are transformed in a certain manner, they are unchanged by the transformation. For instance, a function which is unchanged by reflection across the \(y\)-axis is called an **even function**. Such a function has a graph which is symmetric with respect to the \(y\)-axis and will satisfy the equation \(f(-x) = f(x)\). For instance, the functions \(f(x) = x^2, x^6, \frac{1}{1+x^2}\) area all even functions.
A function whose graph is symmetric with respect to the origin is called an **odd function**. Such a function satisfies the equation $-f(-x) = f(x)$, or equivalently, $f(-x) = -f(x)$. For example, the functions $f(x) = x, x^3, \frac{x}{1+x^2}$ are all odd functions.

In contrast with the use of these terms as applied to numbers, a function can be neither even or odd (in fact, most are neither).

*[Example: p. 197, #18, 19]*

**A vertical stretch transformation** amplifies the height of the entire graph of the function. This has the effect of increasing the outputs of the function by some constant factor of $k$ in order to obtain the transformed function. Symbolically, if $y = f(x)$ is the given function and $k$ the transformation constant, then the transformed function has the form $y = g(x) = k \cdot f(x)$. The situation we describe here holds when $k > 1$. If $0 < k < 1$, then instead of a vertical amplification, the graph is transformed by a vertical compression: the outputs are smaller by a factor of $k$. (If $k = 1$, then there is no transformation!) When $k$ is negative, the change in sign of the transformation factor has the added effect of reflecting the graph across the x-axis.

*[Examples: p. 204, #1, 4]*
A **horizontal stretch transformation** compresses the width of the entire graph of the function. This has the effect of *decreasing* the inputs of the function by some constant factor of $k$ in order to obtain the transformed function. Notice again the difference from the previous situation.

Symbolically, if $y = f(x)$ is the given function and $k$ the transformation constant, then the transformed function has the form $y = g(x) = f(kx)$. The situation we describe here holds when $k > 1$. If $0 < k < 1$, then instead of a horizontal compression, the graph is transformed by a horizontal stretch: compression by a small factor $k$ corresponds to stretching by its reciprocal $1/k$. (Again, if $k = 1$, then there is no transformation.) When $k$ is negative, the change in sign of the transformation factor has the added effect of reflecting the graph across the $y$-axis.

*Examples: p. 211, #5*

The challenge is to incorporate these ideas in combinations of many transformations, and in applications to real situations.

*Examples: p. 205, #19, 21; p. 211, #15, 21*