Using the Maximin Principle

Under the maximin principle, it is easy to see that Rose should choose $a$, making her worst-case payoff 0. Colin’s similar rationality as a player induces him to play (under the same principle) $B$, which causes Rose to receive only the third-best payoff for her (of four) in the game matrix.

\[
\begin{array}{cc}
A & B \\
 a & 5 & 0 \\
b & 4 & 3 \\
\end{array}
\]

What if Rose could outsmart Colin by choosing $b$ sometimes (even though it is not her maximin strategy)? Wouldn’t that cause her to receive a positive payoff against Colin’s choice of $B$ whenever she played $b$? True, but if Colin could infer the pattern with which Rose made these choices, he might be able to thwart her attempts to get a positive payoff.

What if she made those choices at random, choosing $a$ if a coin toss falls heads and $b$ if it falls tails? Colin would have no chance at outmaneuvering her. We call this a mixed (or probabilistic) strategy.
Since half the time, Rose picks \(a\) and half the time she picks \(b\), then her **expected payoff** against Colin’s choice of \(A\) would be \(\frac{1}{2}(5) + \frac{1}{2}(-4) = \frac{1}{2}\). That is, half the time she wins 5 and half the time she loses 4, so on average she wins \(\frac{1}{2}\) on each play.

Against Colin’s choice of \(B\), her expected payoff is \(\frac{1}{2}(0) + \frac{1}{2}(3) = \frac{3}{2}\).

It should be clear that, even though Colin cannot anticipate what strategy Rose will pick, he can perform this same analysis to find that it is better for him to choose \(A\) than \(B\) if Rose picks the mixed strategy \((\frac{1}{2}a, \frac{1}{2}b)\). (Even Rose’s use of a mixed strategy does not prevent Colin from acting rationally.)

Notice that Rose’s pure strategies (pick \(a\) only, or pick \(b\) only) can be seen as mixtures of the form \((1a, 0b)\) or \((0a, 1b)\).

If Rose plays according to the maximin principle and allows for the use of mixed strategies, the \((\frac{1}{2}a, \frac{1}{2}b)\) strategy **does better for her than either pure strategy** \(a\) or \(b\).
Still, Rose can do better than the \((\frac{1}{2}a, \frac{1}{2}b)\) mixed strategy. If she plays the different mixed strategy \((\frac{7}{12}a, \frac{5}{12}b)\), her expected payoffs against both of Colin’s choices is \(\frac{5}{4}\).

We can show that the strategy \((\frac{7}{12}a, \frac{5}{12}b)\) is Rose’s best choice of mixed strategy under the maximin principle: no other mixed (or pure) strategy provides for a higher worst-case expected payoff. So Rose can always expect to win a payoff of at least \(\frac{5}{4}\) whenever she plays the strategy \((\frac{7}{12}a, \frac{5}{12}b)\). Since she can guarantee this payoff for herself, the value of the game for Rose is \(\frac{5}{4}\).

Of course, there is no reason why Colin cannot also use mixed strategies. A similar analysis for him dictates that he play the mixed strategy \((\frac{1}{4}A, \frac{3}{4}B)\), for this will guarantee that his loss is no more than \(\frac{5}{4}\) no matter which pure strategy Rose plays. We call this Colin’s minimax strategy (for Colin to maximize his minimum payoff, he needs to minimize the maximum payoff to Rose). Note that the value of the game to Colin is \(-\frac{5}{4}\), exactly the opposite of that to Rose.
In general, these ideas formed the grounds of a fundamental theorem in game theory proved in 1928 by John von Neumann:

**The Minimax Theorem:** In every two-player zero-sum game in which there are only finitely many strategies available to the players, there is a maximin (mixed or pure) strategy for Rose, a minimax strategy for Colin, and a number $v$, the value of the game, so that when Rose plays her maximin strategy, she guarantees for her a payoff of at least $v$ regardless of what Colin plays, and when Colin plays his minimax strategy, he guarantees for himself a payoff of at least $-v$ regardless of what Rose plays.

In the special important case of the $2\times2$ game, we can be more explicit.
The $2\times 2$ Minimax Theorem: In any $2\times 2$ game there are two possibilities:

- **Either one or both of the players has a dominant strategy.** In this case, the dominant strategy is the maximin strategy for Rose or the minimax strategy for Colin; further, the optimal (maximin or minimax) strategy for the other player is the better corresponding pure strategy for him or her, and the value of the game is the outcome when the players make these choices.

- **Neither player has a dominant strategy.** In such a case, let $a$ be a largest entry in the game matrix (there may be more than one). By switching rows or columns if necessary, we can assume that the matrix has the form

\[
\begin{array}{c|c}
I & II \\
\hline
I & a & b \\
II & c & d \\
\end{array}
\]

Then entry $d$ must be the next largest entry and the maximin strategy for Rose is $(x \cdot I, [1 - x] \cdot II)$, the minimax strategy for Colin is $(y \cdot I, [1 - y] \cdot II)$, and the value of the game is $v$, where

\[
x = \frac{d \cdot c}{(a \cdot b) + (d \cdot c)}, \quad y = \frac{d \cdot b}{(a \cdot b) + (d \cdot c)}
\]

\[
v = \frac{ad \cdot bc}{(a \cdot b) + (d \cdot c)}
\]
This sort of analysis can sometimes be extended to games in which one of the players has more than two pure strategies available, by *elimination of dominated strategies*. Indeed, we may find that one of the strategies a player possesses may be dominated by a *mixture* of some other strategies, which also allows us to eliminate it from consideration.