Graphical Methods

For games in which one of the players has two strategies and the other has more than two, an effective method for finding the optimal maximin/minimax mixed strategies is provided by two types of graphical method. In such cases, if neither player has a dominant pure strategy (else the game is easy to solve), it turns out that the optimal strategy for the player with more than two strategies always simplifies to a mixture of just two of these strategies. Once the appropriate pair of strategies has been identified, we can eliminate the others and reduce the analysis to that of a 2 × 2 game.

The Frontier Method

The first of these is called the frontier method, and is discussed in Hamburger (p. 62–65). It assumes the perspective of the player that has more than two pure strategies to choose from; for the sake of discussion, let us say this is Rose.

We will use the game matrix

\[
\begin{array}{cc}
A & B \\
\hline
a & \begin{array}{c}
0 \\
2
\end{array} \\
b & \begin{array}{c}
1 \\
0
\end{array} \\
c & \begin{array}{c}
1 \\
2
\end{array} \\
d & \begin{array}{c}
3 \\
3
\end{array}
\end{array}
\]

\[
a = (0, 2) \\
b = (1, 0) \\
c = (1, 2) \\
d = (3, 3)
\]
We begin by representing each of Rose’s strategies – a, b, c, d – as a point in a coordinate plane in which the x-axis represents payoffs vs. A and the y-axis represents payoffs vs. B.

Notice that a dominates b for Rose; this is shown in the graph by the fact that a is both to the right of b (a is better than b vs. A) and above b (a is better than b vs. B).
Next, observe that points on the line segment joining \( a \) to \( c \) represent the various mixtures of the strategies \( a \) and \( c \): the point exactly halfway between them represents the mixture \((\frac{1}{2}a, \frac{1}{2}c)\) with expected payoffs \((\frac{1}{2} \cdot 0 + \frac{1}{2} \cdot 1, \frac{1}{2} \cdot 2 + \frac{1}{2} \cdot (-2)) = (\frac{1}{2}, 0)\). The point \( \frac{3}{4} \) of the way from \( a \) to \( c \) is the mixture \((\frac{1}{4}a, \frac{3}{4}c)\) with expected payoff \((\frac{1}{4} \cdot 0 + \frac{3}{4} \cdot 1, \frac{1}{4} \cdot 2 + \frac{3}{4} \cdot (-2)) = (\frac{3}{4}, -1)\). Similarly, the points on the segment joining \( a \) with \( d \) represent the mixtures of those two strategies.

As \( p \) increases from 0 to 1, the mixture \((p \cdot a, [1 - p] \cdot c)\) corresponds to points increasingly closer to \( a \) and further from \( c \). In particular, not only does \( a \) dominate \( b \), but so does every mixture of \( a \) and \( c \) that uses \( a \) with probability \( p > \frac{1}{2} \).
Because strategies that lie to the left and below other strategies are dominated, we need only concern ourselves with strategies on the “northeast frontier” of the points in the graph.

Finally, observe that whatever strategy (pure or mixed) Rose plays, Colin decides the payoff to Rose by either choosing $A$, in which case she receives as payoff the $x$-coordinate of the point in the graph, or he plays $B$, in which case she receives the $y$-coordinate value. It follows that her payoff is independent of what Colin chooses only when both coordinates are equal, namely when it lies on the dotted line $y = x$. Since the segment joining $a$ and $c$ is the only one intersecting the dotted line, Rose’s optimal mixture must be one involving $a$ and $c$. 
The game can therefore be simplified by ignoring strategies $b$ and $d$, neither of which are used in the optimal mixture.

The same sort of analysis can be performed when Colin is the player with more than two pure strategies, except that the direction of domination is reversed: for Colin, strategies are dominated when they lie above and to the right of other strategies. It is the southwest frontier that is of concern to Colin. Its intersection with the diagonal dotted line identifies the pair of strategies whose mixture is optimal for him.
The Envelope Method

This second graphical method is discussed in Straffin, p. 16-18. *The analysis takes the perspective of the player with only two pure strategies*. For our purposes, we use the same example as above, but analyze now from Colin’s point of view.

\[
\begin{array}{c|cc}
 & A & B \\
\hline
a & 0 & 2 \\
b & 1 & 0 \\
c & 1 & 2 \\
d & 3 & 3 \\
\end{array}
\]

This graph takes for the horizontal axis the line segment from 0 to 1 representing Colin’s mixture variable \(y\). (He is trying to find the right mixture \((y \cdot A, [1-y] \cdot B)\) for him to play.) We erect two identical vertical axes at either end of the segment to measure the payoffs (to Rose) from the matrix. At the left end of the graph, \(y = 0\) and Colin is playing \(B\), so we mark the values of the payoffs against \(B\) from the matrix on this axis. At the right end of the graph, \(y = 1\) and Colin is playing \(A\), so we mark the values of the payoffs against \(A\) on
this axis. Then we connect line segments from the points on the left axis to points on the right axis when they represent payoffs from the same row.

Notice that for any value of $y$ between 0 and 1, the points on these line segments above that value of $y$ represent the expected payoffs (to Rose) when she plays the corresponding row against Colin’s mixture ($y \cdot A, [1 - y] \cdot B$). For instance, if $y = \frac{1}{2}$, then Colin plays the mixture ($\frac{1}{2}A, \frac{1}{2}B$) and Rose’s payoffs when she plays $a, b, c, d$ against this mixture are the values halfway between the corresponding payoffs for the pure strategies $A$ and $B$; these are the heights of the points on the segments for $a, b, c, d$ above the point $y = \frac{1}{2}$.
Colin wants to minimize the maximum payoff Rose receives, so he observes that her maximum payoffs lie on the **upper envelope** of the segments in the graph (depicted below in bold).

Thus, he wants to use the value of \( y \) that corresponds to the lowest point on this envelope. It is clear from the graph that this point is the intersection of the segments for \( a \) and \( c \), so once again we see that Colin should act as though Rose has only the two strategies \( a \) and \( c \); the game can be simplified to \( 2 \times 2 \) form.