Cooperative game theory: voting systems

We now turn our attention away from competitive games to consider cooperative games, situations in which a number of players – usually a large number! – make simultaneous decisions that determine an outcome which affects the whole body of actors in the same way. Voting is the preeminent example of such cooperative games.

The simplest such situations involve yes-no voting, or the process of passing a motion before a deliberative body. The motion passes if a sufficient number of yes votes are made, or if the right combination of players vote in favor of the motion.

A collection of voters whose yes votes are sufficient to pass the motion is called a winning coalition.
There are a number of different ways in which a yes-no voting system can be arranged:

- the simplest is just to list all the possible winning coalitions, or perhaps only the minimal winning coalitions (a winning coalition is minimal if no proper subset is also a winning coalition) in the case that the system is monotonic (that is, that adding yes votes to a winning coalition always produces a winning coalition) [e.g., US Federal system for legislation];
- next simplest is the common situation in which voters are assigned a number of votes, or alternately, a single vote that carries a particular weight (such a system is called a weighted voting system), and then a quota is specified that determines the smallest number of votes required to pass the motion [e.g., EEC];
- or the system may involve a combination of the above two characteristics [e.g., UN Security Council].

Many voting systems that are not obviously weighted voting systems can sometimes be reformulated as weighted voting systems. However, it is not always possible to formulate all voting systems as weighted voting systems. The way to see this is to study some of the properties that all weighted voting systems must satisfy.
Swap robustness

A voting system is called swap robust if whenever $X$ and $Y$ are two winning coalitions, and $x$ is in $X$ and $y$ is in $Y$ (but $x$ is not in $Y$ and $y$ is not in $X$), then swapping the voters (sending $x$ to $Y$ and $y$ to $X$) will create two different coalitions at least one of which must remain a winning coalition.

Swap robustness is a property of weighted voting systems, as we shall now see.

Theorem. Weighted voting systems are swap robust.

Proof. Suppose $X$ and $Y$ are some pair of winning coalitions in a weighted voting system. Suppose that swapping $x$ out of $X$ and replacing this voter with $y$ creates the coalition $X'$, and that swapping $y$ out of $Y$ and replacing this voter with $x$ creates the coalition $Y'$. If voters $x$ and $y$ carry the same weight, then $X'$ has the same weight as $X$ had, and $Y'$ has the same weight as $Y$ had. Thus, both new coalitions are winning coalitions.

If $x$ and $y$ have different weights, then one of $x$ or $y$ weighs more than the other. Suppose $x$ is the voter with greater weight. Then $Y'$ has greater weight than $Y$, so it must still be a winning coalition. //