Sexagesimal numeration

**Numeration** [L. numerus = number] signifies the way in which numbers are represented in writing. In our familiar decimal, or base 10, numeration, numbers are represented by means of just ten symbols, the digits 0, 1, 2, 3, 4, 5, 6, 7, 8, 9. It is their relative position in a numeral that determines its value. For instance, we distinguish between the numbers 247 and 724 because

\[
247 = 2 \cdot 100 + 4 \cdot 10 + 7 = 2 \cdot 10^2 + 4 \cdot 10^1 + 7 \cdot 10^0
\]

\[
724 = 7 \cdot 100 + 2 \cdot 10 + 4 = 7 \cdot 10^2 + 2 \cdot 10^1 + 4 \cdot 10^0
\]

This is why we call this numeration decimal *position*al notation: the position of the digits represent successive powers of 10, with the $10^0$, or ones, place at the rightmost position.

Babylonian **sexagesimal** [L. sexagesimus = sixtieth] numeration behaves similarly. In this system, the numbers 0, 1, 2, … , 58, 59 are conceived as single-digit numbers. Ideally, we should have 60 separate symbols for each of them. But clearly this would be unwieldy. Instead, we continue to use our familiar notation for these numbers, with the exception that we will always write 00, 01, 02, … , 09 with a leading 0.
This way, every “single-digit” sexagesimal number will be represented with a pair of decimal digits, e.g., decimal 2 = sexagesimal 02, decimal 37 = sexagesimal 37.

We then order the digits of sexagesimal numbers with their ones digit at the right, the 60s place to its immediate left, the 60^2s place to its left, and so on, just as in decimal notation. We will highlight the fact that we are using a new notation by separating sexagesimal digits with the “pipe” symbol, |. So, for example,

\[
01|40 = 1 \cdot 60^1 + 40 \cdot 60^0 = 60 + 40 = 100
\]
\[
10|10 = 10 \cdot 60^1 + 10 \cdot 60^0 = 600 + 10 = 610
\]
\[
02|35|51 = 2 \cdot 60^2 + 35 \cdot 60^1 + 51 \cdot 60^0
= 7200 + 2100 + 51 = 9351
\]

Pencil-and-paper arithmetic can be performed using sexagesimal notation with the same techniques that we use with our familiar decimal numeration. This is not to say that Babylonian calculators employed these procedures, but it does point out the similarities in the underlying concepts of arithmetic in both systems of numeration.
For instance, addition and subtraction of sexagesimals can be performed by the same methods we learn in grade school, provided we remember that "carrying" and "borrowing" involve units of 60 instead of 10. Study these two examples:

\[
\begin{array}{c}
01 & 01 \\
03 & 23 & 50 \\
+ 01 & 45 & 25 \\
\hline
05 & 09 & 15 \\
\end{array}
\quad \begin{array}{c}
01 & 59 & 75 \\
\hline
02 & 00 & 15 \\
\hline
25 & 40 \\
\hline
01 & 34 & 35 \\
\end{array}
\]

Multiplication can also be carried out with grade-school methods, but in practice, extensive multiplication tables were used to help with these computations:

\[
\begin{array}{c}
12 & 14 & 36 \\
\times \quad 02 & 20 \\
\hline
12 & 00 \\
04 & 40 \\
\hline
04 & 00 \\
01 & 12 \\
28 \\
\hline
24 \\
\hline
28 & 34 & 04 & 00 \\
\end{array}
\]

\[
20 \times 36 = 720 \\
20 \times 14 = 280 \\
20 \times 12 = 240 \\
2 \times 36 = 72 \\
2 \times 14 = 28 \\
2 \times 12 = 24 \\
\]
In the same way that in decimal numeration, digits to the right of the decimal point denote fractional values (numbers smaller than 1), sexagesimal numeration can denote fractional numbers by placing digits to the right of a “sexagesimal point”. Since we already separate sexagesimal digits with a pipe symbol, we will use a double pipe (||) to indicate our “sexagesimal point”. Thus,

\[
05||40|30 = 5 \cdot 60^0 + 40 \cdot 60^{-1} + 30 \cdot 60^{-2}
= 5 + \frac{40}{60} + \frac{30}{60^2}
= 5 \cdot 60^2 + 40 \cdot 60 + 30
= \frac{20430}{3600}
= 5 \frac{27}{40}
= 5.675
\]
Finally, division of sexagesimal numbers was handled in a special way. Rather than attempt a "long division" computation of, say, 03|35|48 by 12, it is recast as a multiplication: we compute instead 1/12 of 03|35|48. Since the number 1/12 is the reciprocal of 12, namely that number which when multiplied by 12 gives 1, the Babylonian calculator would consult extensive tables of reciprocal numbers, and find that the reciprocal of 12 is 00||05 (note that 12 times 5 is 60, then shift the "sexagesimal point" one place to the left!). So our division computation corresponds to multiplying 03|35|48 by 00||05. (Answer?)

Vestiges of sexagesimal numeration are left with us today in how we reckon angle measure and time measure.