Continued fractions

We are most familiar with the method of representing numbers in *decimal* form, all the more since calculators use this representation almost exclusively. However, we are also aware that ancient civilizations used other means to represent numbers: for instance, Babylonian astronomers (and their descendants) favored *sexagesimal* representation, and Mayan calendar makers used a form of *base-20* numeration. In addition, modern society is highly dependent on computers, whose electronic brains naturally operate with *binary* representations of numbers.

There is yet another system for the representation of numbers, used since ancient times. Instead of focusing on methods for expressing larger and larger numbers by means of the relative position of digital symbols in places determined the powers of a fixed base (as all the systems above do), it excels at producing accurate approximations to any real number solely by means of integer quantities.

Recall that a **real number** is any number, positive, negative, or 0, which can be used to quantify some measurement.
A convenient model for the real numbers is the **number line**: mark any two points on a line and identify one as the number 0 and the other as 1, and this establishes a **unit** of length (the segment between the points). Continue marking points off in either direction, each a unit apart, and the corresponding numbers for these points comprise numbers we call **integers**. Beginning at 0 and moving along the line in the direction of 1, the integers can be listed as 1, 2, 3, 4,..., all of which are **positive**; moving along the line in the opposite direction from 0 gives −1, −2, −3, −4,..., the **negative** integers.

Points between the integers correspond to numbers with fractional parts, whose decimal representations include digits after the decimal point. For instance, the number line point corresponding to the number 4.25 is, by virtue of its decimal representation, $\frac{25}{100} = \frac{1}{4}$ of the way from 4 to 5 on the line. There are some points, like the one $\frac{1}{3}$ of the way between 16 and 17, whose decimal representations do not terminate: 16.3333... .
We learn that all rational numbers, namely numbers which can be represented as fractional quantities of the form $\frac{a}{b}$ where $a$ and $b$ are integers ($b \neq 0$ since division by 0 is impossible), have decimal representations which either terminate (as 4.25) or continue forever, with some block of one or more digits that repeats (as 16.3333…). In particular, we note that all integers are rational numbers, since for example, 57 has the fractional representation $\frac{57}{1}$.

Additionally, there are real numbers which are not rational. Examples of such irrational numbers are $\sqrt{2} = 1.14142135623…$ and $\pi = 3.1415926535…$. They also have non-termination decimal expansions, but there are no repeating patterns of digits in these expansions. Every real number is either rational or irrational (and every rational or irrational number is real).

We use real numbers to quantify measurements in the real world. As a result, precision is a primary goal, which we achieve by trying to keep sufficiently many decimal places of accuracy in the measurements we take.
Another way to strive for precision is to use sufficiently close rational number approximations to quantify our measurements. This was the only method available to civilizations like the Egyptians' who did not possess a positional numeration system like decimal or sexagesimal numbers.

Let us illustrate this with an example. In our decimal numeration, we find that the number of synodic months in one tropical year is 12.368266392… months per year. We recognize that further accuracy than 9 places may be possible, but are not likely to be necessary.

In the absence of decimal notation, we can do the following: consider this as a measurement of the tropical year in units of synodic months. We first note that the year is greater than 12 but less than 13 months, so note the integer 12 as the first rational approximation for our measurement. We see that the remainder of the measurement is a fraction of a month; what fraction? We partition a month into an integer number of parts, just as many as are needed so that the desired remainder can be expressed in that many parts.
In our case, 0.368... of a month requires dividing the month into more than 2 parts but fewer than 3, since \( \frac{1}{0.368266392...} = 2.715425613... \), so our next, and closer, approximation to the measurement is 12 months and one of 2 parts. i.e., \( 12 \frac{1}{2} = \frac{25}{2} \) months.

We can repeat this process as much as necessary: we have divided the month into 2 parts, but really, it should have been 2.715... parts. We can now measure the number of parts using the same method as before: it is 2 whole parts and a fraction of a part: but what fraction? Partition the third part into a whole number of parts of parts, just as many as are needed so that the desired remainder can be expressed in that many parts. So, 0.715... of a part requires dividing a part into more than 1 but fewer than 2 parts: \( \frac{1}{0.715425613...} = 1.397769322... \)

Our next, and even closer approximation becomes 12 months, 2 parts, and 1 parts of parts, or

\[
12 \frac{1}{2+\frac{1}{3}} = 12 \frac{1}{3} = \frac{37}{3} \text{ months.}
\]
Notice that at each stage of the process, we peel off the whole number of parts to place deeper into the denominator of the resulting **continued fraction**, and use the remaining fractional part to determine the number of parts needed at the next level, by taking the reciprocal (dividing it into 1): 0.397... has reciprocal 2.514..., so the next level of approximation gives the continued fraction

\[
12 \frac{1}{2+ \frac{1}{1+ \frac{1}{2}}} = 12 \frac{1}{2 + \frac{2}{3}} = 12 \frac{3}{8} = \frac{99}{8} \text{ months,}
\]

and 0.514... has reciprocal 1.945..., so the next level of approximation is

\[
12 \frac{1}{2+ \frac{1}{1+ \frac{1}{2+ \frac{1}{1+ \frac{1}{3}}}}} = 12 \frac{1}{2 + \frac{1}{2 + \frac{1}{3}}} = 12 \frac{4}{11} = \frac{136}{11}
\]

Since .0945... has reciprocal 1.057..., one more level produces
Continued fractions

\[
12 \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{1+\frac{1}{1}}}}} = 12 \frac{1}{2+\frac{1}{1+\frac{1}{2+\frac{1}{2}}}} = 12 \frac{1}{1+\frac{5}{7}} = 12 \frac{7}{19} = \frac{235}{19}
\]

At this point we have a very nice rational approximation \((235/19)\) to the number of synodic months in a tropical year. This is precisely the approximation that produced the Metonic cycle: 235 months in 19 years! An earlier approximation above points to an 8-year cycle of 99 months used in some ancient lunisolar calendars.