Derivatives

If $x_0$ is an input value for the function $f(x)$ and $y_0 = f(x_0)$ is its corresponding output value, then the derivative of the function at $x_0$ is the value of the (instantaneous) rate of change of $f(x)$ at $x_0$; or the slope of the tangent line to the graph of $f(x)$ at the point $(x_0, y_0)$ (that is, the slope of the curve at the point).

There are two standard notations used for the derivative:

• *Leibniz notation*, suggesting the connection with slope computations (difference of outputs over difference of inputs)—
  \[ \frac{d}{dx}[f(x)], \text{ or } \frac{df}{dx}, \text{ or simply } \frac{dy}{dx} \]

• *Legendre notation* (the “derived” output)— $f'(x)$

Like average rates of change, instantaneous rates of change (derivatives) are measured in units of output per units of input.
The derivative \( \frac{df}{dx} \) at \( x = x_0 \) (also written \( f^{[x_0]}(x_0) \)) measures the instantaneous rate of change of the function at \( x = x_0 \). The derivative can be used to approximate the actual change in the output of the function from \( x = x_0 \) to \( x = x_0 + 1 \). That is,

\[
\frac{f^{[x_0]}(x_0) \cdot \frac{f(x_0 + 1) - f(x_0)}{(x_0 + 1) - 1}}{x_0} = f(x_0 + 1) - f(x_0)
\]

It follows that

\[
f(x_0 + 1) = f(x_0) + f^{[x_0]}(x_0).
\]

The **percentage rate of change** of a function \( f(x) \) at a particular input \( x = x_0 \) is the ratio of the rate of change there to the value of the function there (expressed as a percent): \( \frac{f^{[x_0]}(x_0)}{f(x_0)} \). A percentage rate of change is measured in of % points per unit of input.