Accumulated change from rates of change

Suppose that \( r(t) \) measures a rate of change with respect to a time variable \( t \). For example, say that for a two-hour period \((0 \leq t \leq 2)\), we have \( r(t) = 62 \) miles per hour, describing the rate of change in the distance traveled by a driver on a certain highway (i.e., the speed of the vehicle). The total accumulated change in distance traveled by the driver is therefore \((2 \text{ hr}) \cdot (62 \text{ mi/hr}) = 124 \text{ mi}\). Note that graphically, we can interpret this accumulated change as the area of the rectangle formed by the graph of the function \( r(t) = 62 \) over the interval \( 0 \leq t \leq 2 \).

If \( r(t) \) is a linear function, then the speed varies over the interval in question. Suppose that a driver leaves the highway, breaking along an exit ramp from a speed of 60 mph to 24 mph in 10 seconds. It makes sense to convert these speeds to units of ft/sec: 60 mph = 88 ft/sec; and 24 mph = 35.2 ft/sec. Here, \( r(t) \) is a decreasing linear function: \( r(0) = 88 \) to \( r(10) = 35.2 \). The accumulated change in distance over this 10-second period is the same as the change in distance experienced by a vehicle traveling at the (constant) average speed of 61.6 ft/sec for 30 seconds, that is,
Once again, it is possible to represent this graphically as the area of the trapezoid formed by the graph of the linear function \( r(t) \) over the interval \( 0 \leq t \leq 10 \).

These examples point to a more general principle: the accumulated change in a quantity whose rate of change is measured by the function \( f(x) \) over a certain interval of values of \( x \) is equal to the area under the graph of the function \( f(x) \) and over this interval on the \( x \)-axis.

We must interpret area as a signed quantity in this context: if the rate function is negative, so is the associated accumulated change, so area below the \( x \)-axis is to be considered as negative for the purposes of measuring accumulated change.

When the regions are simple rectangles or trapezoids, the area is straightforward to calculate. But when the rate function is a general curve, the region whose area measures the accumulated change can be very complicated to compute exactly. We can always simplify the problem by approximating the area.
Approximating areas

The simplest method of approximating area under a curve is to sample a number of points equally spaced along the input interval so as to dissect the input interval into $n$ subintervals; denote the input values that determine these subintervals by $x_0, x_1, \ldots, x_n$.

The height of the curve at $x_0$ is $f(x_0)$, and we can to build a rectangle having this height over the first subinterval $x_0 \leq x \leq x_1$. The area of this rectangle approximates the area under the curve $f(x)$ over this first subinterval. Similarly, a rectangle of height $f(x_1)$ can be built over the second subinterval $x_1 \leq x \leq x_2$ to approximate the area under the curve $f(x)$ over this second subinterval. Continuing in this fashion, we obtain $n$ rectangles, called left rectangles (since their heights correspond to the heights of the function at their left edges), and their total area approximates the area under $f(x)$ over the entire interval in question. When $n$ is large, the error of the approximation becomes small.
A similar approximation by right rectangles uses $f(x_1)$ as the height over the first subinterval, $f(x_2)$ as the height over the second subinterval, and so on: we use the heights corresponding to the right edges of the subintervals.

If the function $f(x)$ is concave down, then the left rectangle approximation always underestimates the area under the curve, and the right rectangle approximation always overestimates the area under the curve. The reverse is true if $f(x)$ is concave up.

Often, a better approximation is the midpoint rectangle approximation: use $f(x_1)$ as the height over the first subinterval where $x_1$ is the midpoint of the first subinterval $x_0 \leq x \leq x_1$, $f(x_2)$ as the height over the second subinterval where $x_2$ is the midpoint of the second subinterval $x_1 \leq x \leq x_2$, and so on.
Count data

Rate of change functions are meant to be continuous, and accumulated change computations for such functions are found by computing an area under the corresponding curve.

Functions that record discrete counts of a quantity, however, are only approximately modeled by continuous functions; the accumulated values of count data are found not by computing area under the curve, but simply by adding the appropriate counts.
Another method, the \textbf{trapezoidal} approximation, uses trapezoids instead of rectangles. On the first subinterval, we replace the curve joining \((x_0, f(x_0))\) and \((x_1, f(x_1))\) with the line segment joining them and approximate the area under the curve with the area of the resulting trapezoid. Repeating this over each subinterval, and adding the resulting areas gives a very good approximation for many cases.