The definite integral

Recall how we used a number of average rate of change calculations (slopes of secant lines) over tinier and tinier input intervals to get better and better approximations of the instantaneous rate of change at a point (slope of the tangent line).

We have also observed that increasing the number of rectangles (or trapezoids) used in an approximation of the accumulated change in a rate function (the total area) employs narrower and narrower rectangles (or trapezoids) and leads to better and better approximations of the accumulated change (area under the curve).
More specifically, suppose we use \( n \) midpoint rectangles to approximate the accumulated change corresponding to a rate function \( f(x) \) for values of \( x \) from \( x = a \) to \( x = b \).

Then we dissect the interval \( a \leq x \leq b \) into subintervals at the points

\[
(a =) x_0, x_1, x_2, \ldots, x_n (= b)
\]

where the distance between consecutive points is the same, namely, one \( n \)th of the total length of the interval:

\[
\square x = \frac{b - a}{n}.
\]

In the first subinterval, we choose the midpoint \( \bar{x}_1 \) to determine the height \( f(\bar{x}_1) \) of the first rectangle; the area of the rectangle is thus \( f(\bar{x}_1) \cdot \square x \). In the second subinterval, we choose the midpoint \( \bar{x}_2 \) to determine the height \( f(\bar{x}_2) \) of the second rectangle; its area is \( f(\bar{x}_2) \cdot \square x \).

Similarly, we find the areas of the remaining rectangles and add to estimate the total area:

\[
f(\bar{x}_1) \cdot \square x + f(\bar{x}_2) \cdot \square x + \cdots + f(\bar{x}_n) \cdot \square x
\]
This sum is often abbreviated with the notation

\[ \sum_{i=1}^{n} f(x_i) \cdot \Delta x \]

Knowing that the approximation improves by increasing the number of rectangles (and thereby decreasing the widths of each rectangle), we will obtain the exact value of the total area by evaluating the limit of this approximation as \( n \) approaches infinity:

\[ \text{Area} = \lim_{n \to \infty} \sum_{i=1}^{n} f(x_i) \cdot \Delta x \]

In English, this says that the area under the curve is exactly equal to the limiting value of the sum of the midpoint rectangles as the number of rectangles used increases to infinity.
This idea does not depend on the fact that we used midpoint rectangles; it works equally well if we use left or right rectangles, or trapezoids. We define the value of this limit to be the **definite integral** of the function \( f(x) \) over the interval \( a \leq x \leq b \), and we introduce the following notation (due to Leibniz):

\[
\int_{a}^{b} f(x) \, dx = \lim_{n \to \infty} \left( \frac{b-a}{n} \sum_{i=1}^{n} f(x_i) \cdot \Delta x \right)
\]

(Leibniz chose the integral sign \( \int \) to be reminiscent of the summation sign \( \sum \); both are versions of a stylized \( S \).)

In practice, these limits can be calculated by repeatedly computing approximations with larger and larger values of \( n \), stopping when the accuracy of the approximation exceeds the desired level of accuracy in the answer.

[TI-83: use special program PRGM NUMINTG]