Logistic functions

Many quantities display exponential growth in their early stages, but as their environment does not allow unlimited growth, the rate of increase must eventually lessen and the quantity begins to level off (at a rate opposite to how it grew), never exceeding a certain limiting value $L$. This is known as logistic growth.

Logistic decay is characterized by increasingly steep decay from a positive level $L$ that eventually slows as the quantity approaches 0.

Logistic function models have formulas like

$$f(x) = \frac{L}{1 + Ae^{-Bx}}$$

where $L$ is (as above) the positive limiting value. The sign of the parameter $B$ indicates whether the function is describing logistic growth ($B > 0$) or decay ($B < 0$). (We discuss the significance of the parameter $A$ below.)

Logistic functions have characteristic S-shaped graphs, lying between two horizontal asymptotes, the $x$-axis and the line $y = L$. 
A logistic growth curve is always increasing, from near 0 to near \( L \), while a logistic decay curve is always decreasing, from near \( L \) to near 0. In both cases, \( 0 < f(x) < L \).

The shape of the curve changes about the central point of symmetry, the point at which \( f'(x) = \frac{1}{2} L \). To one side of this point, the curve is **concave up** while on the other side it is **concave down**. This point is also called an **inflection point**, since the **concavity** of the curve changes here. Solving the above equation for \( x \) shows that the inflection point is located where \( x = \frac{\ln A}{B} \).

Logistic regression can be used to fit a logistic function to data that exhibit this type of behavior.

**[TI-83: STAT CALC B]**

**Logistic** \(<x\text{-list}>\), \(<y\text{-list}>\), \(<Y\text{-variable}>\)

produces the logistic regression model]
Limits

A central idea in the development of calculus is the concept of a limit. The fundamental concepts of derivative and integral both refer to this idea.

If the output values of a function $f$ get closer and closer to the number $L$ as the inputs get closer and closer to the input $x = a$ and do so while remaining smaller than $a$, then we say that $L$ is the limit as $x$ approaches $a$ from below, and we write

$$
\lim_{x \to \mathbf{a}^-} f(x) = L.
$$

Similarly, if $f(x)$ gets closer to the number $L$ as the inputs get closer to the input $x = a$ and does so while always remaining larger than $a$, then we say that $L$ is the limit as $x$ approaches $a$ from above, and we write

$$
\lim_{x \to \mathbf{a}^+} f(x) = L.
$$

If we wish to evaluate the limit as $x$ approaches $a$ without regard to whether $x$ is larger or smaller than $a$, then we write simply

$$
\lim_{x \to a} f(x) = L.
$$
It is important to note that the value of $L$ need not always be the same as $f(a)$. For instance, if

$$f(x) = \frac{x^2 - 4}{x - 2},$$

then $\lim_{x \to 2} f(x) = 4$ despite the fact $f(2) \neq 4$. Indeed, $f$ is undefined at $x = 2$.

As another example, let $g(x) = \frac{1}{x}$; then we can write

$$\lim_{x \to 0^-} g(x) = -\infty \quad \text{and} \quad \lim_{x \to 0^+} g(x) = +\infty,$$

which nicely describes the difference in the behavior of the function near $x = 0$.

A function is discontinuous at a point $x = a$ whenever the limiting value there disagrees with the function value there, i.e., when

$$\lim_{x \to a} f(x) \neq f(a)$$

The two examples above describe different types of discontinuities ($f$ is discontinuous at $x = 2$, and $g$ is discontinuous at $x = 0$).
A function’s **end behavior** can be described by determining the limiting value(s) of the function at **infinity**:

\[
\lim_{x \to \infty} f(x) = L_+ \quad \text{and} \quad \lim_{x \to -\infty} f(x) = L_-
\]

identify the values to which the outputs approach as their inputs are taken larger and larger (in a positive sense – to the right along the graph, or in a negative sense – to the left). For instance, if \( f \) is a logistic function, these quantities tell where the horizontal asymptotes lie: their equations will have the form \( y = L_+ \) and \( y = L_- \).