A few more Exercises for Section 13.4 (gradient and directional derivative)

(I) Let \( f(x, y) = \cos(xy) + xe^y \).

(a) Find \( \nabla f(x, y) \).

(b) Find an equation of the plane tangent to the surface \( z = f(x, y) \) at the point \( (1, 0, 2) \).

(II) On a certain mountain the elevation \( z \) above a point \( (x, y) \) in the horizontal \( xy \)-plane at sea level is \( z = 2500 - 3x^2 - 4y^2 \) ft. The positive x-axis points east and the positive y-axis points north. Suppose that a climber is at the point \( (15, -10, 1425) \).

(a) If the climber walks due west, will the climber ascend or descend? At what rate?

(b) If the climber walks southeast, will the climber ascend or descend? At what rate?

(c) In what direction should the climber walk to travel a level path?

(III) Suppose that the depth of the ocean at \( (x, y) \) is \( h(x, y) = 2x^2 + 3y^3 \) kilometers. If a ship sails from the point \( (-1, 2) \) in the direction indicated by the vector \( \langle 4, 1 \rangle \), is the water getting shallower or deeper? At what rate? (Assume that \( x \) and \( y \) are measured in kilometers.)

(IV) Captain Kork is in trouble. His ship, the Exitprize, is too near the sun. The temperature of the hull is \( T(x, y, z) = e^{-x^2-2y^2-3z^2} \) when the Exitprize is located at \( (x, y, z) \). \( (x, y, z) \) are measured in meters.) The Exitprize is currently at \( (1, 1, 1) \).

(a) In what direction should the Exitprize travel to decrease the temperature of its hull most rapidly?

(b) If the ship travels at \( 8 \) meters per second, how fast will the temperature decrease if the ship moves in the direction found in part (a) ?

(c) Unfortunately, the hull will crack if it cools at a rate greater than \( 14 \) degrees per second. Assuming the same speed as in (b), describe the set of directions in which the ship can travel safely to cool its hull.

(V) (a) Describe the direction in which the directional derivative of a function is maximal.

(b) Describe the direction in which the directional derivative of a function is minimal.

(c) Describe the direction in which the directional derivative of a function is zero.