\[ \frac{dx}{dt} = ax + by \]
\[ \frac{dy}{dt} = cx + dy \]

In the beginning we looked for straight line solutions.

Look back at how this worked: what we did and why it works.

Why were straight line solutions important?
- We know how they behave.
- There is a clear connection between \((x, y)\) and \((dx/dt, dy/dt)\).
- We can write down a solution formula that stays on that line.

Reinterpret this into matrix-vector context:
\[
\begin{pmatrix} dx/dt \\ dy/dt \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]

Find solutions by looking for eigenvectors of the matrix so that
\[
\frac{dx}{dt} = \frac{dy}{dt} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}
\]

Find E-values of \(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\) & corresponding E-vector:
- \(\lambda\) and the corresponding e-vector \(\begin{pmatrix} x \\ y \end{pmatrix}\) such that
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}
\]

The idea is that if \(\frac{dx}{dt} = \begin{pmatrix} x \\ y \end{pmatrix}\) and if \(\begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}\) then the \(\frac{d}{dt}\) operation must result in a multiplication by \(\lambda\).

So by \(a = b, b = c \Rightarrow a = c\) then finding the derivative is the same as finding \(\lambda\), which leads us back to \(e^{\lambda t}\).

If \(\begin{pmatrix} x \\ y \end{pmatrix}\) is an e-vector for e-value lambda, then so is every scalar multiple of \(\begin{pmatrix} x \\ y \end{pmatrix}\): once you have an eigenvector a scalar multiple is also an eigenvector, such as \(C\begin{pmatrix} x \\ y \end{pmatrix}\) or \(e^{\lambda t}\begin{pmatrix} x \\ y \end{pmatrix}\) or \(Ce^{\lambda t}\begin{pmatrix} x \\ y \end{pmatrix}\).

All we need to do is create an eigenvector so that \(\frac{d}{dt}\) of it is the same as \(\lambda\), which is why this works.

Therefore \(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ce^{\lambda t}\begin{pmatrix} x \\ y \end{pmatrix}\) is a solution (eigenvector) -- so find eigenvector, then say this is a solution because we have established a solid chain of equalities.

The point is that (1) differentiating \(\frac{d}{dt}\) causes multiplication by \(\lambda\) because of the \(e^{\lambda t}\) term and (2) the \(A\begin{pmatrix} x \\ y \end{pmatrix}\) operation causes a multiplication by lambda because \(\begin{pmatrix} x(t) \\ y(t) \end{pmatrix}\) is an e-vector of \(A\) for eigenvalue \(\lambda\) for all \(t\).

This is why the straight line solutions were important: they showed us the way.
what we really want is a general solution so we can write down a solution for an arbitrary initial condition
we need two $\lambda$ so we can make a linear combination
to find a general solution we needed two straight line solutions (translating into two eigenvalues each with their own
eigenvector: both are solutions for the same reasons

$$
\begin{pmatrix}
x_1(t) \\
y_1(t)
\end{pmatrix} = Ce^{\lambda_1 t} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \quad \text{and} \quad 
\begin{pmatrix}
x_2(t) \\
y_2(t)
\end{pmatrix} = De^{\lambda_2 t} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
$$

how do we get a general solution from these parts? add them
we must ensure that when we add them then the new creature is also a solution
how? simple addition: when differentiating a sum, differentiate the pieces and add. matrix mappings are linear, and so is the differentiation.
we need to know that $\begin{pmatrix} x_1(t) \\ y_1(t) \end{pmatrix} + \begin{pmatrix} x_2(t) \\ y_2(t) \end{pmatrix}$ solves the system, and the eigenvectors must be linearly independent so that adding them gives a solution for every possible initial condition, not just those on the line.
If we have two different eigenvectors with two different eigenvalues they must be linearly independent. If that is the case, then therefore, for a fixed $t$, we can reach any initial condition. Our general solution is :

$$
\begin{pmatrix}
x_1(t) \\
y_1(t)
\end{pmatrix} = Ce^{\lambda_1 t} \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} 
+ +
\begin{pmatrix}
x_2(t) \\
y_2(t)
\end{pmatrix} = De^{\lambda_2 t} \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}
$$

Now that we have a general solution, we want to step now to what happens if we have no straight line solution: that is, if there is no real $\lambda$
so where do things become different?

how did we find $\lambda$? finding a $\lambda$ necessitates that $\det \begin{pmatrix} a - \lambda & b \\ c & d - \lambda \end{pmatrix} = 0$, so $(a - \lambda)(d - \lambda) - bc = 0$ which is a
quadratic in lambda which means that we may not have a real $\lambda$ and hence no straight line solution.
There are always 2 solutions if we allow complex lambdas (which opens up a whole new universe)

$$
\lambda = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \quad \text{(not the same abc as the abcd in the matrix above)} \quad \text{so if the discriminant is positive then we get two}
$$

real $\lambda$; if zero then one solution; if negative then two complex $\lambda$.
so $\lambda = a + ib$ (a and b are neither the previous a and b, nor related to any previous a and b)
we are now in the complex “universe” and have to deal with a potentially imaginary part
claim: everything works the same
writing down $\lambda$ gets us everything just as before whether $\lambda$ has an imaginary part or not
so what we want is $\frac{d}{dt} \begin{pmatrix} x \\ y \end{pmatrix} = A \begin{pmatrix} x \\ y \end{pmatrix}$, and we have found a (possibly) complex $\lambda = a + ib$ so that $A \begin{pmatrix} x \\ y \end{pmatrix} = \lambda \begin{pmatrix} x \\ y \end{pmatrix}$ for

some (possibly) complex eigenvector $\begin{pmatrix} x \\ y \end{pmatrix}$, ie x and y in the eigenvector may be complex
for this to be true, the eigenvector must have an imaginary part whenever the $\lambda$ has an imaginary part
now what?
we have a complex eigenvalue and eigenvector. everything works the same, even with complex numbers
so $\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = Ce^{it} \begin{pmatrix} x \\ y \end{pmatrix}$ will still be a solution to the system
remember, $\begin{pmatrix} x \\ y \end{pmatrix}$ is an eigenvector for the eigenvalue lambda

Earlier we needed two solutions to cover every initial condition
however, in this case we only need one solution to cover all the initial conditions. we don’t need the second one, and it’s longer that way -- there is another way
here’s what we’re going to do:
the vector $\begin{pmatrix} x \\ y \end{pmatrix}$ can be written as $\begin{pmatrix} x_r \\ y_r \end{pmatrix} + i \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ (the imaginary term cannot be the zero vector)

(all the numbers in $A$ are real, as are the x/y coordinates: we must have an imaginary part in both the eigenvector and $\lambda$)
here’s the bonus: the real and imaginary components are linearly independent and so we have access to a general solution through one $\lambda$

we will see that each of $e^{it} \begin{pmatrix} x_r \\ y_r \end{pmatrix}$ and $e^{it} \begin{pmatrix} x_i \\ y_i \end{pmatrix}$ are solutions and linearly independent

how do we prove this?
we want to show that these are solutions
example:
\[
\begin{align*}
\frac{dx}{dt} &= -2x - 3y \\
\frac{dy}{dt} &= 3x - 2y
\end{align*}
\]
\[
\begin{pmatrix} \frac{dx}{dt} \\ \frac{dy}{dt} \end{pmatrix} = \begin{pmatrix} -2 & -3 \\ 3 & -2 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}
\]
we are trying to find eigenvalue and eigenvector, so find $\det \begin{pmatrix} -2 - \lambda & -3 \\ 3 & -2 - \lambda \end{pmatrix} = 0$

\[
(-2 - \lambda)(-2 - \lambda) - (-3)(3) = 0
\]
$\lambda^2 + 4\lambda + 13 = 0$
\[
\lambda = \frac{-4 \pm \sqrt{4^2 - 4(1)(13)}}{2(1)} = \frac{-4 \pm \sqrt{16 - 52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = -2 \pm 3i
\]
applying quadratic formula, $\lambda = -2 \pm 3i$ so we have two $\lambda$
using $\lambda = -2 + 3i$, find an eigenvector (note: only using the plus 3i, not both at once here)

$$\begin{pmatrix}-2 & -3 \\ 3 & -2\end{pmatrix}\begin{pmatrix}x \\ y\end{pmatrix} = (-2 + 3i)\begin{pmatrix}x \\ y\end{pmatrix}$$

so

$$-2x - 3y = (-2 + 3i)x$$

so

$$3x - 2y = (-2 + 3i)y$$

simplifying gives

$$y = -ix$$

$$x = iy$$

are they compatible? if $y = -ix$ then it is true that $x = iy$ sign change is due to $i^2$ and

$$\begin{pmatrix}x \\ y\end{pmatrix} = \begin{pmatrix}i \\ 1\end{pmatrix}$$

so now

$$\begin{pmatrix}x(t) \\ y(t)\end{pmatrix} = e^{(-2+3i)t}\begin{pmatrix}i \\ 1\end{pmatrix}$$

but we can’t deal with complex numbers in the real world so now we assume that writing this as a real piece plus $i$ another real piece then we have two linearly independent solutions and we are done

now the task is to rewrite this as $V(t) = V_r(t) + iV_i(t)$ ($V =$ vector shorthand)

how do we do this?

To do this, we can use the famous Euler’s formula which says that $e^{a+ib} = e^a e^{ib} = e^a (\cos(b) + i\sin(b))$

[implication: $e^{i\pi} + 1 = 0$: world’s five most famous numbers in one equation and nothing else!]

so how do we use this identity to rewrite our formula in some useful way?

$$e^{(-2+3i)t}\begin{pmatrix}i \\ 1\end{pmatrix}$$

$$= e^{-2t+3it}\begin{pmatrix}i \\ 1\end{pmatrix}$$

$$= e^{-2t}\left(\cos(3t) + i\sin(3t)\right)\begin{pmatrix}i \\ 1\end{pmatrix}$$

$$\left(\left(e^{-2t}\cos(3t) + i\sin(3t)\right)(i) \right)$$

$$\left(e^{-2t}\cos(3t) + i\sin(3t)\right)\begin{pmatrix}i \\ 1\end{pmatrix}$$

$$= \left(e^{-2t}\cos(3t)i + e^{-2t}i^2\sin(3t)\right)$$

$$\left(e^{-2t}\cos(3t) + e^{-2t}\sin(3t)\right)$$

$$= \left(-e^{-2t}\sin(3t) + ie^{-2t}\cos(3t)\right)$$

$$\left(e^{-2t}\cos(3t) + ie^{-2t}\sin(3t)\right)$$

$$= \left(-e^{-2t}\sin(3t) + i\left(e^{-2t}\cos(3t)\right)\right)$$

$$\left(e^{-2t}\cos(3t) + i\left(e^{-2t}\sin(3t)\right)\right)$$

= we end up with a real vector + $i$ real vector

so complex lambda leads to complex vector leads to two independent pieces for the general solution egad!