ENCYCLOPÉDIE OU *DICTIONNAIRE* RAISONNÉ
DES SCIENCES, DES ARTS ET DES MÉTIERS

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Heads or tails, (*analysis of chances*.) This game which is well known, & which has no need of definition, will provide us the following reflections. One asks how much are the odds that one will bring *heads* in playing two successive tosses. The answer that one will find in all authors, & following the ordinary principles, is this one here: There are four combinations,

First toss.	Second toss.
Heads.	*Heads*.
Tails.	*Heads*.
Heads.	*Tails*.
Tails.	*Tails*.

Of these four combinations one alone makes a loss, & three make wins; the odds are therefore 3 against 1 in favor of the player who casts the piece. If he wagered on three tosses, one will find eight combinations of which one alone makes a loss, & seven make wins; thus the odds will be 7 against 1. *See* **Combinaison** & **Avantage**. However is this quite correct? For in order to take here only the case of two tosses, is it not necessary to reduce to one the two combinations which give *heads* on the first toss? For as soon as *heads* comes one time, the game is finished, & the second toss counts for nothing. So there are properly only three possible combinations:

Heads, first toss.
Tails, *heads*, first & second toss.
Tails, *tails*, first & second toss.

Therefore the odds are 2 against 1. Likewise in the case of three tosses, one will find

Heads.
Tails, heads.
Tails, tails, heads.
Tails, tails, tails.

Therefore the odds are only 3 against 1: this is worthy, it seems me, of the attention of the Calculators, & would go to reform well some unanimously received rules on the games of chance.

Another question. Pierre plays against Paul on this condition, that if Pierre brings *heads* on the first toss, he will pay an ecu[1] to Paul; if *heads* is brought only on the second toss, two ecus; if on the third toss, four, & thus in succession. One finds by the ordinary rules

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[1]An obsolete French coin worth about 3 francs.

(by following the principle that we have just admitted), that the expectation of Paul, & consequently that which he must put into the game is

$$\frac{1+2+4+\&\mathrm{c}}{1+1+1+\&\mathrm{c}}$$

a quantity which is found infinite. However there is not a person who wishes to put into this game a sum at all considerable. One is able to see in the *Mémoires de l'académy de Petersbourg*, Book V. some attempts to resolve this difficulty; but we do not know if one will be satisfied by them; & there is here some scandal which well deserves to occupy the algebraists. That which seems surprising in the solution of this problem, is the infinite quantity that one finds for the expectation of Paul. But one will notice that the expectation of Paul must be equal to the risk of Pierre. So there is only concern to know if the risk of Pierre is infinite, that is to say (following the real notion of infinity) if this risk is such a one that one can always suppose it larger than any assignable finite number. Now as little as one reflects on the question, one will see that this risk is such in fact. For this risk increases with the number of the tosses, as is very evident by the calculus. Now the number of tosses are able to progress & progress indeed to infinity, since by the conditions of game the number is not fixed. Therefore the indefinite number some tosses is one of the reasons which the risk of Pierre found infinite here. *See* **Absent** & **Probabilité**.

According to a very learned geometer with whom I reasoned one day on this matter, the expectation of Paul & his stake can never be infinite, because the wealth of Pierre is not; & because if Pierre has, for example, only 2^{20} ecus of wealth, he must have only 21 tosses, after which he must quit, because Pierre won't be in a state to pay. So the number of the possible tosses is determined, finite, & equal to 21, & one will find that the expectation of Paul is $\frac{2^{21}-1}{22}$. Although this sum is no longer infinite, I doubt that any player would ever wish to give it. So this solution, all ingenious as it is, does not seem at first to resolve the difficulty. However all things examined rightly, it seems to me that one must be satisfied by it. For there is no concern here of the penalty or of the ease that Paul must have to risk the sum in question, the question is what he must give in order to play in an equal game with Pierre; & it is certain that what he must give is the sum above. Paul would be a fool without doubt to give it; but he would be, only because Pierre is a fool also to propose a game where Pierre could lose to him in one minute some immense sums. Now, in order to play with a fool in an equal game, it is necessary to be mad like he. If Pierre playing in a single toss, wagered a million that it will bring *tails*, it would be necessary that each set towards the game a half-million: this is incontestable. There are however only two madmen who are able to play a similar game.

We will remark at this opportunity, that in order to render more complete, & in order to therefore say more customary, the solutions of problems concerning the games, it would be to wish that one could incorporate within the moral considerations, relative, either to the fortune of the players, or to their state, or to their situation, to their same strength (when it concerns some games of commerce), & thus of the rest. It is certain, for example, that two unequally rich men who play at an equal game following the ordinary laws, the one who is the less rich risks more than the other. But all these considerations being nearly impossible to submit to the calculus because of the diversity of the circumstances, one is obligated to set aside, & to resolve the problems mathematically, by supposing moreover the moral circumstances perfectly equal on both sides, or in disregarding them completely. It is then these circumstances, when one comes to pay attention to them, which makes believe the calculation in error, although it is not here. *See* **Avantage**, **Jeu**, **Pari**, &c. (*M. d'Alembert*)