Quadrilaterals

Let the points $A, B, C, D$ be coplanar with no three of collinear. Then a quadrilateral, denoted $\triangle ABCD$, is the union of the four segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$, provided none of the segments meet except at the endpoints. The points $A, B, C, D$ are called the vertices of the quadrilateral, the segments $\overline{AB}, \overline{BC}, \overline{CD}, \overline{DA}$ are its sides, and $\angle ABC, \angle BCD, \angle CDA, \angle DAB$ its angles. Any two of the objects which are listed consecutively above are said to be consecutive, or adjacent, vertices, sides, or angles (also $D$ and $A$ are consecutive vertices, $\overline{DA}$ and $\overline{AB}$ are consecutive sides, and $\angle DAB$ and $\angle ABC$ are consecutive angles). Further, any two of the objects which are not listed consecutively above are said to be opposite. The segments $\overline{AC}, \overline{BD}$ joining opposite vertices are the diagonals of the quadrilateral.

A quadrilateral is convex if its diagonals intersect at a point which is interior to each diagonal. In particular, each vertex is interior to the angle made at its opposite vertex (why?).

Two quadrilaterals $\triangle ABCD$ and $\triangle WXYZ$ are congruent if the one-to-one correspondence $A \mapsto W, B \mapsto X, C \mapsto Y, D \mapsto Z$ of its vertices induces a correspondence of congruent sides and congruent angles. That is, CPCF.
**Theorem [SASAS implies congruence]** If
\( \triangle ABCD \) and \( \triangle WXYZ \) are two quadrilaterals for which three consecutive sides are congruent
\( (AB \cong WX, BC \cong XY, CD \cong YZ) \) and the two included angles are congruent
\( (\angle ABC \cong \angle WXY, \angle BCD \cong \angle XYZ) \), then the quadrilaterals are congruent
\( (\triangle ABCD \cong \triangle WXYZ) \). //

**Theorem [ASASASA implies congruence]** If
\( \triangle ABCD \) and \( \triangle WXYZ \) are two quadrilaterals for which three consecutive angles are congruent
\( (\angle ABC \cong \angle WXY, \angle BCD \cong \angle XYZ, \angle CDA \cong \angle YZW) \) and the two included sides are congruent
\( (BC \cong XY, CD \cong YZ) \), then the quadrilaterals are congruent
\( (\triangle ABCD \cong \triangle WXYZ) \). //

**Theorem [SASAA implies congruence]** If
\( \triangle ABCD \) and \( \triangle WXYZ \) are two quadrilaterals for which two consecutive sides are congruent
\( (AB \cong WX, BC \cong XY) \) and three consecutive angles are congruent, including the one contained by the congruent sides
\( (\angle ABC \cong \angle WXY, \angle BCD \cong \angle XYZ, \angle CDA \cong \angle YZW) \), then the quadrilaterals are congruent
\( (\triangle ABCD \cong \triangle WXYZ) \). //

**Theorem [SASSSS implies congruence]** If
\( \triangle ABCD \) and \( \triangle WXYZ \) are two quadrilaterals for which all four pairs of sides are congruent and
some pair of angles is congruent, then the quadrilaterals are congruent. //
A rectangle is a quadrilateral whose angles are all right angles. In the geometry whose axioms we have assembled at present, called absolute geometry, it is not even clear that a rectangle can exist! The existence of a rectangle in absolute geometry is a historically important question, one taken up by the eighteenth century Jesuit geometer Giovanni Girolamo Saccheri in his influential 1733 work *Euclides ab omni naevo vindicatus* (*Euclid, Freed of Every Flaw*). In it, he defined a quadrilateral $\Diamond ABCD$ with base $\overline{AB}$, whose adjacent base angles $\angle A$ and $\angle B$ are right, and whose legs, the adjacent sides $\overline{DA}$ and $\overline{BC}$, are congruent (Figure 3.59, p. 186); such we call a Saccheri quadrilateral. The side $\overline{CD}$ opposite the base is called the summit, and its adjacent angles $\angle C$ and $\angle D$ are the summit angles of the quadrilateral.

While in Euclidean geometry it is clear that both summit angles of a Saccheri quadrilateral are right and that the quadrilateral is a rectangle — Saccheri labored to find such a proof, without success — this is not provable in absolute geometry. To see this, notice that one can exhibit a Saccheri quadrilateral in the Poincaré model for geometry whose summit angles are acute! (See Figure 3.60, p. 186.) Also, in spherical geometry one can exhibit a Saccheri quadrilateral whose summit angles are obtuse.
**Lemma** A Saccheri quadrilateral is convex. //

**Theorem** The summit angles of a Saccheri quadrilateral are congruent. //

**Corollary** The diagonals of a Saccheri quadrilateral are congruent. //

**Corollary** The line through the midpoints of the base and summit of a Saccheri quadrilateral is the perpendicular bisector of both segments. //

**Corollary** If the summit angles of a Saccheri quadrilateral are right, then the quadrilateral is a rectangle and its base and summit are congruent. //

Saccheri attempted to prove that his quadrilateral was a rectangle using an argument by contradiction. Since the summit angles are congruent, they must be either both acute, both obtuse or both right; if he could reach a contradiction under the first two hypotheses, the third must therefore be true. This led to his formulation of the **hypothesis of the acute angle** (that the summit angles are acute), the **hypothesis of the obtuse angle** (that the summit angles are obtuse), and the **hypothesis of the right angle** (that the summit angles are right). Significantly, he was able to prove the
**Theorem** The hypothesis of the obtuse angle is not valid in absolute geometry. //

But to his frustration, he was not able to reach the desired contradiction for the hypothesis of the acute angle! (And we know now that it would be impossible to do so.) He was able to obtain partial results:

**Theorem** The summit of a Saccheri quadrilateral has length at least as large as its base. //

**Corollary** The segment joining the midpoints of two sides of a triangle has length no greater than half the length of the third side of the triangle. //

In 1766 Johann Heinrich Lambert (first to give a proof that $\pi$ is an irrational number) wrote another influential work on the foundations of Euclidean geometry, *Theorie der Parallellinien* (*Theory of Parallel Lines*). In this work, he dealt with a quadrilateral having three right angles, which we now call a **Lambert quadrilateral**. It was his aim at first to prove that the fourth angle was also right, but when this was unsuccessful, he proceeded to work out some of the consequences of the strange geometry that followed from the hypothesis of the acute angle.